DIFFERENTIAL INEQUALITIES FOR A FINITE SYSTEM OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, some basic fractional differential inequalities for a finite system of a IVP of hybrid fractional differential equations with linear perturbations of second type are proved. The existence results for maximal and minimal solutions are also obtained and finally a comparison theorem or the hybrid fractional differential has been established.

Keywords and phrases: Hybrid differential equation; Differential inequalities; Comparison result; Extremal solutions.

AMS (MOS) Subject Classifications: 34K10.

1 Introduction

Given a closed and bounded interval $J = [t_0, t_0+a]$ in $\mathbb{R}$, $\mathbb{R}$ being the real line, and given a real number $0 < q < 1$, consider a finite system of perturbed fractional differential equation (in short FDE)

$$\left.\begin{aligned} &D^q[x(t) - f(t, x(t))] = g(t, x(t)), \ t \in J \\ &[x(t) - f(t, x(t))](t - t_0)^{q-1}\Big|_{t=t_0} = X^o, \end{aligned}\right\} \tag{1.1}$$

where D^q is the fractional derivative of non-integer order q, $0 < q < 1$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a *solution* of the FDE (1.1) we mean a function $x \in C(J, \mathbb{R})$ satisfying

(i) the map $t \mapsto x - f(t, x)$ is continuous for each $x \in \mathbb{R}^n$, and

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(ii) $D^q[x(t) - f(t, x(t))]$ exists and satisfies (1.1) on J.

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The FDE (1.1) is a hybrid non-integer order fractional differential equation with a linear perturbation of second type and include the following system of FDE,

$$\left.\begin{aligned} D^q x(t) = g(t, x(t)),\ t \in J \\ x(t)(t - t_0)^{q-1}\Big|_{t=t_0} = x^o, \end{aligned}\right\} \tag{1.2}$$

as a special case. A systematic account of different types of perturbed differential equations is given in Dhage [1]. The FDE (1.2) has been studied for different aspects of the solution by several authors in the literature. The details of fractional differential equations and their applications are given in Kilbas *et. al* [5] and Podlubny [6]. In this paper, we discuss some basic differential inequalities for the hybrid (1.1) on J under suitable conditions.

2 Strict and Nonstrict Inequalities

We need the following definitions in what follows.

Definition 2.1. *A function $f(t, x)$ is said to be quasi-monotone increasing in $x \in \mathbb{R}^n$ if and only if $x < y$, $x_i < y_i$ $i = 1, 2, \ldots, n$; then $f_i(t, x) < f_i(t, y)$ for each $i = 1, 2, \ldots, n$ and for each $t \in J$.*

Definition 2.2. *A function $f(t, x)$ is said to be quasi-monotone nondecreasing in $x \in \mathbb{R}^n$ if $x \leq y$, $x_i \leq y_i$ $i = 1, 2, \ldots, n$; then $f_i(t, x) \leq f_i(t, y)$ for each $i = 1, 2, \ldots, n$ and for each $t \in J$.*

We consider the following hypotheses in the sequel.

(A_0) The mapping $x \mapsto x - f(t, x)$ is quasi-monotone increasing for each $t \in J$, and

(B_0) The mapping $x \mapsto g(t, x)$ is quasi-monotone nondecreasing for each $t \in J$.

Theorem 2.1. *Let $x, y \in C_p(J, \mathbb{R}^n)$ be two locally holder continuous with an exponent λq, $0 < \lambda < 1$ and let hypotheses (A_0) and (B_0) hold. Suppose that*

$$\left.\begin{aligned} D^q[x(t) - f(t, x(t)) \leq g(t, x(t)),\ t \in J \\ [x(t) - f(t, x(t))](t - t_0)^{q-1}\Big|_{t=t_0} = X^o, \end{aligned}\right\} \tag{2.1}$$

and

$$\left.\begin{aligned} D^q[y(t) - f(t, y(t))] \geq g(t, y(t)),\ t \in J \\ [y(t) - f(t, y(t))](t - t_0)^{q-1}\Big|_{t=t_0} = Y^o. \end{aligned}\right\} \tag{2.2}$$

If one of the inequalities (2.1) *and* (2.2) *is strict and*

$$X^o < Y^o \tag{2.3}$$

then

$$x(t) < y(t),\ \ t \in J. \tag{2.4}$$

Proof. Suppose that inequality (2.4) is not true. Define

$$X(t) = x(t) - f(t, x(t))$$

and

$$Y(t) = y(t) - f(t, y(t))$$

for each $t \in J$. Then from the contuity of the functions X^o and Y^o it follows that there exist an index j, $1 \leq j \leq n$ and $t_0 \leq t_1 \leq t_0 + a$ such that

$$X_j(t_1) = Y_j(t_1), \quad X_j(t) \leq Y_j(t),\ \ t_0 \leq t_1 < t_{t_0} + a$$

and

$$X_i(t) \leq Y_i(t), \quad i \neq j.$$

Setting

$$M_j(t_1) = 0, \quad M_j(t) \leq 0,\ \ t_0 \leq t \leq t_1,$$

and

$$M_i(t)(t_1) \leq 0, \quad i \neq j.$$

Applying a standard result we obtain,

$$D^q M_j(t_1) \geq 0. \tag{2.5}$$

Now, assuming the strict inequality (2.2, we obtain from hypotheses (A_0) and (B_0) that

$$\begin{aligned} g_j(t, x_1(t_1), \cdots, x_n(t_1)) &\geq D^q\big[x_j(t_1) - f_j(t_1, x_1(t_1), \cdots, x_n(t_1)))\big] \\ &\geq D^q\big[y_j(t_1) - f_j(t_1, y_1(t_1), \cdots, y_n(t_1)))\big] \\ &> g_j(t, y_1(t_1), \cdots, y_n(t_1)) \end{aligned} \tag{2.6}$$

The above relation (2.6) is a contradiction and hence the relation (2.4) holds on J. This completes the proof. □

Next result is a nonstrict inequality for the hybrid FDE (1.1) on J. This result is proved under one sided Lipschitz contrition.

Theorem 2.2. *Assume that the inequalities* (2.1) *and* (2.2) *with nonstrict inequalities and that the hypotheses* (A_0) *and* (B_0) *hold Further suppose that*

$$g_i(t,x) - g_i(t,y) \le L(x_i - y_i),\ x_i \ge y_i \tag{2.7}$$

for each i, $1 \le i \le n$ *and* $L > 0$.

Then,

$$X^o = [x(t) - f(t,x(t))](t-t_0)^{1-q} \le [y(t) - f(t,y(t))](t-t_0)^{1-q} = Y^o \tag{2.8}$$

implies

$$x(t) \le y(t),\ \ t \in J. \tag{2.9}$$

Proof. We set

$$Y_\epsilon(t) = Y(t) + \epsilon\lambda(t) \tag{2.10}$$

fr each $\epsilon > 0$, $\epsilon \in \mathbb{R}^n$, where $\lambda(t) = (t-t_0)^{1-q}E_{q,q}2L(t-t_0)^q$. This shows that

$$Y_\epsilon > Y^o > X^o$$

which yields that

$$Y_\epsilon(t) > Y(t).$$

Now employing the Lipschitz condition,

$$\begin{aligned} D^q\big[y(t) - f(t,y(t))\big] &= D^qY(t) + \epsilon D^q\lambda(t) \\ &\ge g(t,y(t)) + 2\epsilon L\lambda(t) \\ &\ge g(t,y_\epsilon(t)) - L\epsilon\lambda(t) + 2L\epsilon\lambda(t) \\ &> g(t,y_\epsilon(t)). \end{aligned}$$

Here, we have employed the fact that $\lambda(t)$ is a solution of the IVP

$$D^q\lambda(t) = 2L\lambda(t),\ \ \big[\lambda(t) - f(t,\lambda(t))\big](t-t_0)^{1-q}\Big|_{t=t_0} = \Gamma^o$$

with $\Gamma^o = 1$. Now we apply Theorem 2.1 to $Y_\epsilon(t)$ and $X(t)$ to get

$$Y_\epsilon(t) > X(t),\ \ t \in J.$$

When $\epsilon \to 0$, we obtain the desired conclusion. $\square$

The importance of the differential inequalities lies in their applications. Similarly, differential inequalities proved in Theorem 2.1 and 2.2 are very much useful for proving the other aspects for the hybrid FDE (1.1) J. Next, we prove the comparison theorems for FDE (1.1), since comparison theorems are the powerful tools for proving global existence and uniqueness results for differential and integral equations. Hence, differential and integral inequalities have got importance place in the theory of differential and integral equations.

Before stating our comparison result, we list some basic hypotheses concerning the functions involved in the FDE (1.1). These hypotheses are needed for proving the existence theorem for the FDE (1.1). We only state the existence result, because its roof is similar to that of a scalar case treated in Dhage and Mugale [2].

(A_1) There exist constants L and M such that

$$|f(t,x) - f(t,y)| \leq \frac{L|x-y|_n}{M + |x-y|_n}$$

for all $t \in J$, where $|\cdot|_n$ is a norm in $\mathbb{R}^n$. Moreover, we assume that $L \leq M$.

(B_1) The function g is bounded on $J \times \mathbb{R}^n$.

Theorem 2.3 (Existence theorem)**.** *Assume that hypotheses (A_0)-(A_1) and (B_0)-(B_1) hold. Then the hybrid FDE (1.1) admits a solution.*

Theorem 2.4 (Comparison theorem)**.** *Assume that hypotheses $m \in C_p(J, \mathbb{R}^n)$ is locally holder continuous and*

$$D^q[m(t) - f(t, m(t))] \leq g(t, m(t)) \tag{2.11}$$

for all $t \in J$. Let $r(t)$ be the maximal solution of the IVP

$$\left.\begin{aligned} &D^q[u(t) - f(t,u(t))] = g(t,u(t)),\ t \in J \\ &[u(t) - f(t,u(t))](t-t_0)^{q-1}\Big|_{t=t_0} = U^o, \end{aligned}\right\} \tag{2.12}$$

existing on J such that

$$M^o = [m(t) - f(t,m(t))](t-t_0)^{1-q}\Big|_{t=t_0} \leq U^o. \tag{2.13}$$

Then, we have

$$m(t) \leq r(t), \quad t \in J. \tag{2.14}$$

Proof. From the notion of a maximal solution $r(t)$, it is enough to rove that

$$m(t) \leq r(t,\epsilon), \quad t \in J \tag{2.15}$$

where $r(t,\epsilon)$ is any solution of the hybrid FDE

$$\left.\begin{aligned} D^q[u(t) - f(t,u(t))] &= g(t,u(t)) + \epsilon,\ t \in J \\ [u(t) - f(t,u(t))](t-t_0)^{q-1}\Big|_{t=t_0} &= U^o + \epsilon, \end{aligned}\right\} \tag{2.16}$$

for all $t \in J$, where $\epsilon > 0$ is small number in $\mathbb{R}^n$.

Now the expression in (2.14) yields

$$\begin{aligned} D^q[u(t) - f(t,u(t))] &= g(t,u(t)) + \epsilon \\ &> g(t,u(t)) \end{aligned}$$

Applying strict inequality formulated in Theorem in 2.1, we obtain

$$m(t) \leq r(t,\epsilon), \quad t \in J. \tag{2.17}$$

Since,

$$\lim_{\epsilon \to 0} r(t,\epsilon) = r(t)$$

uniformly on J. Hence taking limit as $\epsilon \to 0$ in (2.16) yields (2.12). This completes the proof. □

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