On Generalised B- Manifolds

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Abstract

GF- Structure Manifolds were defined and studied by Prof. Duggal, K.L, (1971), Prof. Mishra, R.S. (1974) and many other geometers.

In this paper, I have defined real vector space with GF- Structure and B-Scalar product and certain results have been proved. It has been proved that B-Scalar product admits an Orthonormal J-Base .Besides this certain results have also been proved. GF-Structure, Generalised B-manifold has also been studied and certain results on GF-Structure, Generalised B-manifolds, have also been established. Some results on Connections have also been obtained.

On Generalised B-Manifolds

In this paper, we have defined and studied generalised B-Manifolds. Certain interesting theorems have been proved. Some results on connections have also been obtained.

1. Real vector space with GF-structure and B-scalar product.

Let V be a 2n dimensional real vector space with GF-Structure $F^2 = a^2 I$ where I is the identity of the vector space V. It is well known that a scalar product g on V is said to be Hermitian scalar product iff g (FX, FY) =g(X, Y) for all X, $Y \in V$.

Theorem 1: Let V be a 2n dimensional real vector space admitting GF–structure. If h is any GF-bilinear symmetric non degenerate form on V considered as an n dimensional GF-structure Manifold, then h is B-scalar product on the real vector space V.

Proof: Let us consider real vector space V with GF–structure J as an n dimensional complex vector space where aX = -JX for $X \in V$. Let h be a GF-bilinear symmetric non degenerate forms on the GF-structure Manifold V.
Putting h(X, Y) =g(X, Y) –aF(X, Y), for X, $Y \in V$, g and F are bilinear symmetric non degenerate forms on the vector space V. Thus we get
$g(JX, JY) = a^2 g(X,Y)$, $F(JX, JY)= a^2 F(X,Y)$ because of the symmetry of g and F.
Let (V, g, J) be a 2n dimensional vector space V, with a structure J and a B- scalar product g and U be a linear subspace of V. The space U is called non degenerate if the restriction of g on U is a non degenerate form on U. Then U is called degenerate (isotropic) space i.e. there exists a non zero vector $Z \in U^{\perp}$ orthogonal to $U : (U \cap U^{\perp} \neq \{0\})$ .A vector subspace U is called a completely isotropic space, if every vector $Z \in U^{\perp}$ is null vector. The set of the null vectors in V is called isotropic cone. The space U is called holomorphic if JU=U. It is clear that the orthogonal complement $U^{\perp}$ of U is also a holomorphic linear space .We shall call a base $\{X_1, X_2, \ldots, X_n;\ J_{X_1}, \ldots, J_{X_n}\}$ orthonormal J base if $g\left(X_i, J_{X_i}\right) = 0$ and $g(X_i, X_j) = \delta$ ij, where δ ij is Kroneker' s symbol. For this base we have $g\left(J_{X_i}, J_{X_j}\right) = -\delta_{ij}$.

Theorem 2: Any vector space with a GF-structure and a B-scalar product admits an orthonormal J- base.

Proof: Let (V, g, f) be a 2n dimensional real vector space with a GF-structure and a B-scalar product. Let α_1CV be any linear non degenerate holomorphic two dimensional space (non degenerate holomorphic 2-plane) spanned by X, J_X where X is not a null vector. Let us denote by X the vector $g(X,X)^{-1/2}$ X When g(X, X)>0 or $g(J_X,J_X)^{-1/2} J_X$, when g(X, X)<0. Considering vector X_1 such that $X_1 = \lambda X + \gamma J_X, g(X_1, X_1) = 1$

and $g(X_1, J_{X_1}) = 0$. Thus $\{X_1, J_{X_1}\}$ is a unique (in spite of orientation) orthonormal J – base of α_1. Let $\alpha_i, i \in \{2,3,...,n\}$ be a non degenerate holomorphic 2 plane. Every 2-plane $\alpha_j, j \in \{1,2,...,n\}$ admits an orthonormal J base $\{X_J, J_{X_j}\}$. Therefore $\{X_1, X_2, ..., X_n, J_{X_1}, J_{X_2}, ..., J_{X_n}\}$ is an orthonormal J- base of V. Thus from the proof of the theorem (2) it is clear that (V, g, f) is a direct sum of n non- degenerate holomorphic 2 planes. On the other hand, (V, g, J) is a direct sum of two maximal completely isotropic n dimensional linear subspaces U and W. Really let $\{X_1, X_2, ..., X_n, J_{X_1}, J_{X_2}, ..., J_{X_n}\}$ be an orthonormal J-base of V. Then putting $f_i = J_{X_i} - X_i$, $e_i = J_{X_i} + X_i$ we get bases $\{f_1, f_2, ..., f_n\}$, $\{e_1, e_2, \ldots, e_n\}$ of U and W respectively. It is clear that $\{f_1, f_2, ..., f_n, e_1, e_2, \ldots, e_n\}$ is a base of V. we have $g(f_i, f_j)=g(e_i, e_j)=0$, $g(f_i, e_j)= -2\delta_{ij}$. A base like the last one is called skew base [1].

We note that according to theorem (2) the space (V, g, J) is a pseudo- Euclidean vector space with structure (n, n)because (V, g, J) is isometric to the real co-ordinate 2n-dimensional vector space R^{2n} with bilinear form $b_n(X,Y)=\sum(X_iY_j - X_jY_i)$

Where $X=(X^1, X^2,\ldots,X^{2n})$, $Y=(Y^1, Y^2,\ldots,Y^{2n})$ in R^{2n}.

Let (V, J) be a vector space with a GF-structure and let $V_c = V \otimes R^{2n}$

We have $V_c = V^{1,0} + V^{0,1}$ where $V^{1,0} = \{X - aJ_X / X \in V\}$. We can verify directly the following theorem [2].

Theorem 3: A B-scalar product g on (V, J) can be continued uniquely to a GF-symmetric bilinear form g' on V_C such that

- **(a)** g' (X, Y) =0 for $X \in V^{1,0}$ and $Y \in V^{0,1}$
- **(b)** g' (X,X) $\neq 0$ at least for some $X \in V_C$
- **(c)** g' (X,Y) = g' (Y,X), for $X, Y \in V_C$

Conversely every GF-symmetric bilinear form g' on V_C with properties a, b, c is a natural continuation of a B scalar product g on (V, J).

2. Generalised B-Manifold:

Let g be pseudo Riemannian metric on almost GF-Manifold $(M, \Im)$. We shall call g as B-metric, if $g(\Im X, \Im Y) = -g(X, Y)$ for $X, Y \in \therefore (M)$. Evidently, a B-metric g on M defines a B-scalar product g_p in every tangent space T_P (M), $p \in M$ with respect to the GF-structure $\Im_p$ of T_P (M) induced by the almost GF-structure $\Im$ of M. We shall call every almost GF-manifold with a B-metric a generalized B-manifold and let us denote by GB the class of generalised B-manifolds.

Let $M \in GB$ and ∇ be the Levi-Civita connection generated by the metric g of M if $\nabla J = 0$, then M is the known B-Manifold

It is known that every paracompact manifold admits a Riemannian metric. If h is a Riemannian metric on a paracompact almost GF-structure manifold $(M, \Im)$, which is not a Hermitian one, then the metric g with a property $g(X,Y) = h(X,Y) - h(\Im X, \Im Y)$ for all $X, Y \in \therefore (M)$ is a B-metric.

Lemma 1: If $M \in GB$ and $X, Y, Z \in \therefore (M)$, then

(2.1) $g((\nabla_X \Im)Y, Z) = g((\nabla_X \Im)Z, Y)$

(2.2) $g((\nabla_X \Im)Y, JZ) = -g((\nabla_X \Im)Z, \Im Y)$

(2.3) $g((\nabla_X \Im)\Im Y, \Im Z) = g((\nabla_X \Im)Y, Z)$

Where ∇ is the Levi-Civita connection generated by g.

Proof: Since $M \in GB$, so $Zg(\Im_X, Y) = Zg(X, \Im_Y)$ for all $X, Y, Z \in \therefore (M)$. Since ∇ is Levi-Civita connection, the last identity gives (2.1) and since M is an almost GF-manifold, it follows that

(2.4) $(\nabla_X \Im)\Im Y = -\Im(\nabla_X \Im)Y$

The identity (2.2) follows from (2.1) and (2.4). Also (2.3) follows from the condition that g is a B-metric and (2.4).
Now we define a tensorfield ϕ of type (0,3) by the condition

(2.5) $$\phi(X,Y,Z) = g((\nabla_X \Im)Y, Z) + g((\nabla_Y \Im)Z, X) + g((\nabla_Z \Im)X, Y) \text{ for } X,Y,Z \in \therefore (M)$$

Because of (2.1) the tensorfield ϕ is symmetric with respect to any two arguments. By virtue of lemma (1) and relations (2.4) and (2.5), we state the following assertion.

Lemma 2: Let $M \in GB$ and N be the Nijenhuis tensor of $\Im$. Then for all $X,Y,Z \in \therefore (M)$ we have

$$2(g(X,(\nabla_Z \Im)Y) - (\nabla_{\Im Z} \Im)(Y)) = \phi(X,Y,Z) - \phi(X,\Im Y,\Im Z) - g(N(Y,Z),\Im X)$$

Where

$$N(Y,Z) = (\nabla_{\Im Y} \Im)Z - (\nabla_{\Im Z} \Im)Y - \Im(\nabla_Y \Im)Z + \Im(\nabla_Z \Im)Y$$

For $M \in GB$, we shall call M a normal generalized B-manifold if N=0 and let us denote the class of these manifolds by R^{GB}
Using lemma (1) and lemma (2), we can prove the following theorem

Theorem 4: If $M \in R^{GB}$, then $M \in B$ when $\phi = 0$

Because of theorem (4), we have the following result.

Theorem 5: Let $M \in GB$ and $X,Y \in \therefore (M)$. Then the following assertions are equivalent:

(a) $M \in B$
(b) $(\nabla_X \Im)Y = \in (\nabla_Y \Im)X, \in = \pm 1$
(c) $(\nabla_{\Im X} \Im)Y = \lambda \Im(\nabla_X Y), \lambda \in M$
(d) $N_1(X,Y) = (\nabla_{\Im X} \Im)\Im Y - (\nabla_{\Im} Y \Im)\Im X + (\nabla_X \Im)Y - (\nabla_Y \Im)X = 0$
(e) $Q_1(X,Y) = (\nabla_{\Im X} \Im)\Im Y + (\nabla_{\Im Y} \Im)\Im X + (\nabla_X \Im)Y + (\nabla_Y \Im)X = 0$

Proof: Using lemma (1), after some calculations we find the identities.

$$g(N_1(X,Y),\Im Z) + g(N_1(Y,Z),\Im X) - g(N_1(Z,X),\Im Y) = 2(g((\nabla_{\Im X} \Im)Z - (\nabla_{\Im Z} \Im)X, Y) - g((\nabla_Y \Im)X, \Im Z))$$

$$g(Q_1(X,Y),\Im Z) + g(Q_1(Y,Z),\Im X) - g(Q_1(Z,X),\Im Y) = 2(g((\nabla_{\Im Y} \Im)\Im X + (\nabla_X \Im)Y, \Im Z) - g((\nabla_Z \Im)X, \Im Y))$$

by means of these identities, lemma (1) and (2) we establish correctness of the theorem.

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