A MATHEMATICAL COMPUTATION OF HYDRODYNAMICS MODEL WITH VARIABLE COEFFICIENTS IN A UNIFORM RESERVOIR BY USING THE LAX-WENDROFF METHOD

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Thesis Title	A Mathematical Computation of Hydrodynamics Model with Variable Coefficients in a Uniform Reservoir by using the Lax-Wendroff Method
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ABSTRACT

The purposes of this research are to develop mathematical models and numerical methods for approximating water flow directions and pollution levels in a uniform reservoir with non-uniform current. Generally, the pollution levels in a reservoir are measured via data collection at the real site. It is somehow rather complex and the results obtained tentatively deviate from one point of time/place to another. Previous research work applied a mathematical model called "the dispersion model" to estimating the water pollution levels. Nonetheless, the estimation accuracy received is seemingly unsatisfactory, especially, when the water flow is not uniformly distributed.

This research begins with developing a mathematical model which combines two existing mathematical models: a hydrodynamic model and a dispersion model. Using equation, (Ninomiya H. and Onishi K., 1991)

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}[(h+\zeta)u]+\frac{\partial}{\partial y}[(h+\zeta)v]=0,$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0,$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=0,$$

to make the model suitable to uniform reservoirs, the hydrodynamic model is modified by changing its constant coefficient into variable one. It is then converted to a dimensionless form of equations. Firstly the Lax-Wendroff method is subsequently used to estimate the water velocity and elevation. Secondly, the forward differences in time and central difference in space were used in convection-

diffusion equation,

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right).$$

The final result was

$$C_{l,m}^{n+1}=\left(\lambda D+\frac{\Delta t}{2\Delta x}u_{l.m}^{n}\right)C_{l-1,m}^{n}\left(1-2\lambda D-2\lambda D\right)C_{l,m}^{n}+\left(\lambda D-\frac{\Delta t}{2\Delta x}u_{l.m}^{n}\right)C_{l+1,m}^{n}$$
$$+\left(\lambda D+\frac{\Delta t}{2\Delta y}v_{l,m}^{n}\right)C_{l,m-1}^{n}+\left(\lambda D-\frac{\Delta t}{2\Delta y}v_{l,m}^{n}\right)C_{l,m+1}^{n}.$$

The equation combined with the calculated velocity is thus used in the dispersion model to approximate the concentration levels of the pollutants.

The result of this research shows that the proposed model can estimate the water velocity, the elevation, and the flow pattern as well as the concentration of the pollutants in a uniform reservoir at any various time and place. The accuracy of estimation is within the units of centimeters and seconds. In addition, the proposed model can be applied to other water sources having non-uniformly distributed water flows.

Keywords : Hydrodynamic Model with Variable Coefficients, Dispersion Model, Uniform Reservoir, Lax-Wendroff Method, Forward Time, Central Space (FTCS)

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Chunya Sornsri

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List of Abbreviations

Symbol	
m	Metre
s	Second
kg/m^3	Kilogram per cubic metre
$m/\sec$	Metre per second
rad	Radian
rad/s	Radian per second
m/s^2	Metre per square second
$m^{1/2}/s$	Metre square two time part leavings per second

CHAPTER 1

INTRODUCTION

1.1 Water pollution problems

The increases in an industrial occupation is the principal reason for the growth of pollution. Water quality must be protected and maintained for several use, the principal ones being domestic water supply, energy production, industry, agriculture, fish and wildlife. The highest priority use is domestic water supply, with priorities for other uses depending largely on local or reginal conditions and factors. Water pollution can effect humans in many ways, depending on the purpose for which the water resources are to be used. Since it affects human lives, it is health problem.

The term *to pollute* may be defined as to *destroy the purity* of or to *make foul or dirty*. Water pollution may therefore be defined as the alteration of the characteristics of a receiving water body in such a way as to make it unfit for one or more specific uses. To state it another way, pollution refers to the changes in the natural physical, chemical, and biological characteristics of a receiving water caused by the discharge of any material into that water that detracts from beneficial use.

Control of pollution is necessary for the protection of the water environment and the maintenance of acceptable quality in rivers, lakes, reservoirs, streams, estuaries, oceans, and groundwater, The standard, in turn, will depend on the uses to be made of the receiving waters: water supply, fishing-wildlife, industrial, and other uses.

The methods to detect the amount of pollutant both in the air and water mostly are conducted by a field measurement and a mathematical simulation.

1.2 Water quality standards

Water quality standards have generally taken one of two forms: receiving water standards and effluent standards.

1.2.1 Receiving water standards are classified in several categories, according to the highest beneficial use expected.

1.2.2 Effluent standards are limitations which are placed on the effluents and must be limit by each individual source at the point of discharge.

1.3 Water pollution measurement

In water quality modeling, the amount of pollutant in a system is represented by its mass. The utility of concentration is depend on it represents the strength rather than the quality of the pollution. As such it is preferable to use concentration as an indicators of impact on the environment.

1.4 Literature review

Tabuenca (1997: 313-332) developed the mathematical models to simulate pollution in the Bay of Santander. They combined the hydrodynamic and water quality model that provides the velocity field, height and pollutant concentration of the water. Both models are formulated in two-dimensional equations. A case study in simulation of pollution evolution in the Bay of Santander is presented.

Pochai (2006: 755-756) There are many methods to detect and control the amount of water pollution in the water area. In this thesis, the mathematical models are used to assess and control the pollution due to sewage effluent in the water area. The thesis gives the mathematical models of water pollution measurement in one-dimensional and two-dimensional water areas with regular and irregular boundaries. The cases of steady flow and unsteady flow of water current in open and close water areas are considered. The finite element and the finite difference models are used to calculate the water current and the pollutant concentration for variable inputs. These are applicable for the optimal control of water treatment in the system and to achieve minimum cost respectively. The results found that the computer programmes are constructed and used for the water treatment process in both open and close water areas.

Pochai (2008: 803-814) in this research, two mathematical models are used to simulate pollution due to sewage effluent in the uniform reservoir with varied current velocity. The first is the hydrodynamic model that provides the velocity field and elevation of the water flow. The second is the dispersion model that gives the pollutant concentration fields. In the simulating processes, we used the Lax-Wendroff method to system of hydrodynamic model and the forward in time and backward in space technique to the dispersion model. The accuracy of the models is tested by the example.

Pochai (2009: 463-466) A mathematical model is used to simulate the water current and the elevation in a uniform reservoir. A non-linear hydrodynamic model that provides the velocity field and elevation of the water flow is considered. In the simulating process, the Lax-Wendroff technique is used to approximate the solutions. The numerical solution can be the input data for a water-quality model.

1.5 Objectives and Scope of the thesis

A Mathematical model is used to simulate the water current and the elevation in a uniform Reservoir as a Figure 1. The hydrodynamic model provide the velocity and elevation of water. Our research, we focus on the Lax-Wendroff method to approximate the solution of hydrodynamic model with variable coefficients. Those results are needed data for the convection-diffusion equation on the water quality model.

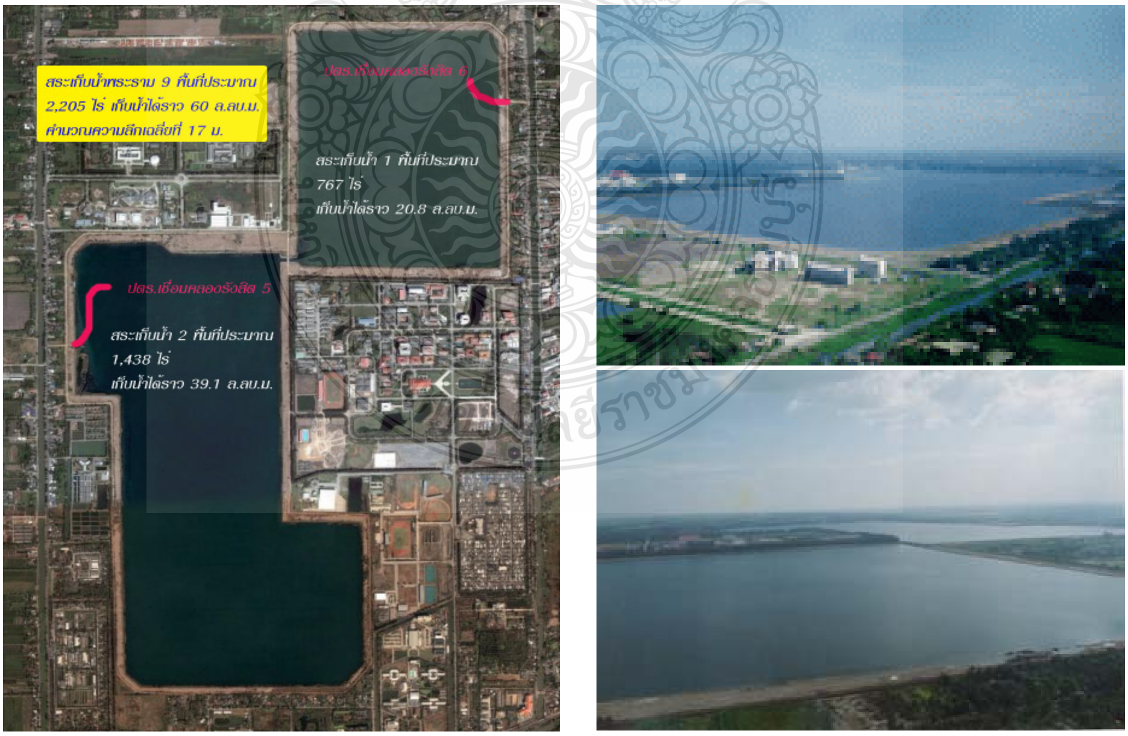


Figure 1 Uniform reservoir *Rama 9 reservoir* (http://www.haii.or.th)

1.6 Plan of the thesis

In this thesis describes the mathematical modelling of the water quality measurement in a uniform reservoir. The first part gives the detail of the basic knowledge of the flow analysis and mathematical modelling for water pollution measurement in two dimensional problems.

The second part gives the computation of the water quality measurement involved the numerical solution of a shallow water equation. We also give the fundamental of the formulation of governing equation that non-dimensional form of shallow water equation in two dimensional water area such as reservoir.

The third part gives the formulation of the unsteady water quality measurement involved the hydrodynamic model in the case of simply domain boundary and uniform bottom topography.
In the simulating processes, we used the Lax-Wendroff method to system of hydrodynamic model and the Forward time/Central space (FTCS) to the dispersion model. The application of the models are illustrated by the examples.

1.7 Expect of result

A couple models for simulation of the velocity, elevation and pollutant concentration in a uniform reservoir is proposed. Since the numerical solutions are good agreement, simple numerical schemes are attractive to use in these models.

CHAPTER 2

BASIC KNOWLEDGE

2.1 Hydrodynamic model

2.1.1 Hydrodynamic model shallow water equations

Horizontal flow of the seawater introduced by the motion of the moon and the sun is called the tidal current. The vertical motion of the sea surfaced is called the tide. The tide is about 12 hours and 25 minutes of period at the place, where the tidal motion is semi-diurnal. The continual motion of the sea water in some definite direction is called the ocean current. In this section, we will give the basic equations which described the tidal current.

We consider an area of the sea with rectangular coordinates $x, y, z(m)$. Let the surface of the mean sea level designate the reference plane $z = 0$. Let $h(x, y)$ be the depth measured from the mean sea level to the sea bed. The elevation from the mean sea level to the temporary sea surface is called the tidal elevation, and it is denoted by $\zeta(x, y, t)$ with the time variable $t(s)$. The total depth of the sea is therefore given by $H = h + \zeta$. We assume that the density of the seawater $\rho(kg/m^3)$ is constant. With the velocity components $u, v(m/\sec)$ of the current to the east and north respectively, the components $U, V(m/\sec)$ of the vertically averaged velocity are expressed by

$$U = \frac{1}{H}\int_{-h}^{\zeta} u dz, \tag{1}$$

$$V = \frac{1}{H}\int_{-h}^{\zeta} v dz. \tag{2}$$

The equation of continuity can then be written as follows Ninomiya H. and Onishi K. (1991)

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (HU)}{\partial x} + \frac{\partial (HV)}{\partial y} = 0. \tag{3}$$

Suppose that, the motion of the sea water, the acceleration in its vertical direction is negligibly small. Based on the assumption that the pressure is hydrostatic, the equations of motion can be given approximately as follows (Ninomiya H. and Onichi K., 1991),

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}-fV+g\frac{\partial \zeta}{\partial x}=\frac{1}{\rho H}\{\frac{\partial}{\partial x}(\mu_\varepsilon H\frac{\partial U}{\partial x})+\frac{\partial}{\partial y}(\mu_\varepsilon H\frac{\partial U}{\partial y})\}$$

$$+\frac{KW^2}{H}\cos\psi-\frac{gU(U^2+V^2)^{\frac{1}{2}}}{HC^2}, \qquad (4)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}-fU+g\frac{\partial \zeta}{\partial y}=\frac{1}{\rho H}\{\frac{\partial}{\partial x}(\mu_\varepsilon H\frac{\partial V}{\partial x})+\frac{\partial}{\partial y}(\mu_\varepsilon H\frac{\partial V}{\partial y})\}$$

$$+\frac{KW^2}{H}\sin\psi-\frac{gV(U^2+V^2)^{\frac{1}{2}}}{HC^2}, \qquad (5)$$

where f is the Coriolis factor, given by $f=2\omega\sin\phi$ with the angular velocity of the terrestrial rotation $\omega=7.292\times10^{-5}(rad/s)$ and the latitude $\phi(rad)$; g is the gravity acceleration $9.81(m/s^2)$, μ_ε is the eddy viscosity, K is the non-dimensional coefficient of a superficial force due to the wind blowing on the surface, W is the wind speed (m/s) 10 meters high above the sea surface, ψ is the angle of the wind direction from the east, and C is the Chezy coefficient $(m^{1/2}/s)$ of the friction on the sea bed. The coefficient are often given by

$$K=\begin{cases}1.0\times10^{-3} \ if \ W\le 5,\\ 1.5\times10^{-3} \ if \ 5<W\le 15,\\ 2.0\times10^{-3} \ if \ 15<W\le 20,\end{cases}$$

and

$$C=\frac{1}{n}h^{1/6} \qquad (6)$$

with the Manning's coefficient of roughness n.

If the shearing stresses, the surface wind and frictional forces are neglected, (Ninomiya H. and Onishi K., 1991), the equations of motion become

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}-fV+g\frac{\partial \zeta}{\partial y}=0, \qquad (7)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}-fU+g\frac{\partial \zeta}{\partial y}=0 \qquad (8)$$

The equations (3), (7) and (8) are called the shallow water equations.

2.1.2 The initial and boundary conditions

There are two kinds of boundary conditions that we are considered. (i) Along the seashore denoted by Γ_V, the normal component of the current velocity is equal to zero:

$$V_n = Un_x + Vn_y = 0 \text{ on } \Gamma_V, \quad (9)$$

where n_x, n_y are the components of the external unit normal to the boundary.

(ii) Along the boundary Γ_ζ, where the region is open to the outer sea, the tidal elevation is prescribed. For the sake of simplicity we consider

$$\zeta(t) = A\sin(\frac{2\pi}{T}t) \text{ on } \Gamma_\zeta, \quad (10)$$

where A is the amplitude of the monochromatic tidal wave (m), and $T = 12 + 5/12\ (hours)$ being its period.

The initial condition in a real sea is not certain. Usually, the calculation starts with calm sea, we suppose the $\zeta = 0, U = V = 0$ at $t = 0$, and in Ω, where Ω is the domain to be analyzed, This technique is called a cold start.

2.1.3 The non-dimensional form of two-dimensional of hydrodynamic model in the uniform reservoir

The continuity and momentum equation are governed the hydrodynamic behavior of the reservoir (Ninomiya H. and Onishi K., 1991). We average the equation over the depth, discarding the term due to the Coriolis force, shearing stresses and surface wind. The well-known two-dimensional shallow water equations are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[(h+\zeta)u] + \frac{\partial}{\partial y}[(h+\zeta)v] = 0, \quad (11)$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \quad (12)$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial y} = 0, \quad (13)$$

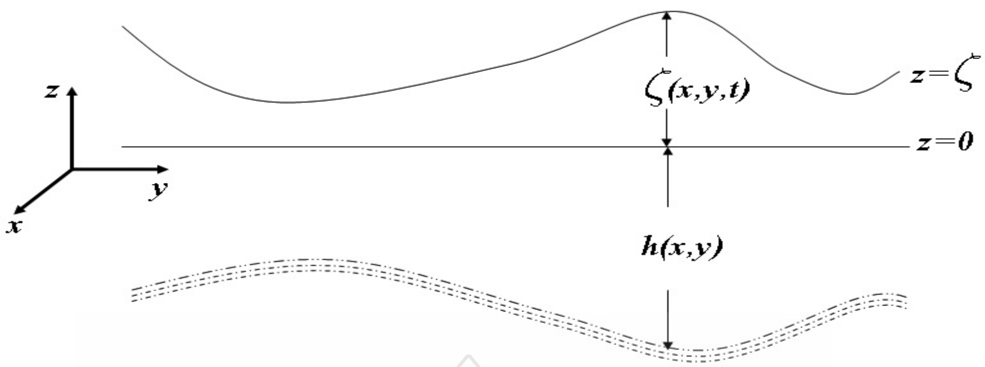


Figure 2 The shallow water system in the reservoir.

where $h(x, y)$ is the depth measured from the mean water level to the bed of the reservoir, $\zeta(x, y, t)$ is the elevation from the mean water level to the temporary water surface or the tidal elevation, g is the acceleration due to gravity, and $u(x, y, t)$ and $v(x, y, t)$ are the velocity components, for all $(x, y) \in [0, l] \times [0, l]$. We now introduce the two-dimensional non-linear shallow water equation with dimensionless form (Pochai, 2008: 803-814) by letting $U = u / \sqrt{gh}, V = v / \sqrt{gh}$, $X = x / l$, $Y = y / l$, $Z = \zeta / h$ and $T = t\sqrt{gh} / l$. We assume that $h + \zeta \cong h$. From equation (11), we obtain

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} hu + \frac{\partial}{\partial y} hv = 0, \tag{14}$$

$$\frac{\partial Zh}{\partial \frac{Tl}{\sqrt{gh}}} + \frac{\partial}{\partial Xl} hU\sqrt{gh} + \frac{\partial}{\partial Yl} hV\sqrt{gh} = 0, \tag{15}$$

$$\frac{h\sqrt{gh}}{l}\frac{\partial Z}{\partial T} + \frac{h\sqrt{gh}}{l}\frac{\partial U}{\partial X} + \frac{h\sqrt{gh}}{l}\frac{\partial V}{\partial Y} = 0, \tag{16}$$

Equation (16) multiply by $\frac{l}{h\sqrt{gh}}$, we get

$$\frac{\partial Z}{\partial T} + \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{17}$$

From equation (12), we can see that

$$\frac{\partial U\sqrt{gh}}{\partial \frac{Tl}{\sqrt{gh}}} + g\frac{\partial Zh}{\partial Xl} = 0, \tag{18}$$

$$\frac{gh}{l}\frac{\partial U}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial X}=0, \tag{19}$$

Equation (19) multiply by $\frac{l}{gh}$, we have

$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=0. \tag{20}$$

From equation (13), we can see that

$$\frac{\partial V\sqrt{gh}}{\partial \frac{Tl}{\sqrt{gh}}}+g\frac{\partial Zh}{\partial Yl}=0, \tag{21}$$

$$\frac{gh}{l}\frac{\partial V}{\partial T}+\frac{gh}{l}\frac{\partial Z}{\partial Y}=0, \tag{22}$$

Equation (22) multiply by $\frac{l}{gh}$, we obtain

$$\frac{\partial V}{\partial T}+\frac{\partial Z}{\partial Y}=0. \tag{23}$$

Assume that the coefficients of $\frac{\partial Z}{\partial X}$ and $\frac{\partial Z}{\partial Y}$ are $(1+XY)$ respectively, where $0\le X,Y\le 1$. Equations (17), (20), (23) become

$$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{24}$$

$$\frac{\partial U}{\partial T}+(1+XY)\frac{\partial Z}{\partial X}=0, \tag{25}$$

$$\frac{\partial V}{\partial T}+(1+XY)\frac{\partial Z}{\partial Y}=0. \tag{26}$$

Initial condition

In $\Omega\times[0,\Im]$ where $\Omega=(0,1)\times(0,1)$ with the initial conditions $Z(X,Y,T)=f(X,Y)$ and $U(X,Y,0)=V(X,Y,0)=0$.

Boundary condition

$Z(0,Y,T)=Z(1,Y,T)=Z(X,0,T)=Z(X,1,T)=0$ at $\partial\Omega$.

2.2 Dispersion model

2.2.1 Convection-diffusion equation

Numerous types of water motion transport matter within natural waters. Wind energy and gravity impart motion to the water that leads to mass transport. In the study within-system motion can be divided into two general categories: convection (advection) and diffusion.

The convection results is unidirectional and does not change the identity of the substance being transported. Convection moves matter from one position in space to another. Simple examples of transport primarily of this type are the flow of water through a lake’s outlet and downstream transport due to flow in a river or estuary.

Diffusion refers to the movement of mass due to random water motion or mixing. Such transport causes the dye patch depicted to spread out and dilute over time with negligible net movement.

The dispersion of the concentration is described by the convection-diffusion equation in an arbitrary domain $\Omega \subseteq \mathbb{R}^n, n = 1, 2, 3$

$$\frac{\partial \Gamma}{\partial t} + \Upsilon \cdot \nabla \Gamma = D \nabla^2 \Gamma, \qquad (27)$$

where

∇	is defined by $\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k,$
∇^2	is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$
$\Gamma(x,t)$	is a concentration terms,
$\Upsilon(x,t)$	is a velocity terms,
D	is a molecular diffusivity.

Both Γ and Υ are function of position x and time t. The domain boundary can be classified into two types, that are Ω_1 with specified concentration and Ω_2 with specified flux of concentration, and total boundary $\partial\Omega$ is $\partial\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \phi$. The boundary condition concerning C are given as follows: the concentration is specified in part of the boundary on Ω_1 ,and the diffusive flux in the exterior normal direction is specified on the rest of the boundary on Ω_2 are

$$C = C_B \text{ on } \Omega_1, \qquad (28)$$

$$-D\frac{\partial C}{\partial n} = T_B \text{ on } \Omega_2. \qquad (29)$$

If the boundary is a non-absorbing or reflexive boundary, then we can put $T_B = 0.$ If a discharge of the mass is instantaneous, the amount of the discharge can be considered in the initial condition $C_B = C_0$ where C_0 is a constant.

2.2.2 The two-dimensional unsteady convection-diffusion equation

The distributed pollutant process satisfies the mass transfer equation, which includes transportation and diffusion. Averaging the equation over the depth, we get the convection-diffusion equation in two-dimensional domain $\Omega \subset \mathbb{R}^2$,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}), \tag{30}$$

where $C(x, y, t)$ is the concentration averaged in depth at the point (x, y) and at time t, D is the diffusion coefficient and $u(x, y, t)$, $v(x, y, t)$ are the velocity component. The initial condition $C(x, y, 0) = g(x, y)$ at $t = 0$ and the boundary condition $\frac{\partial C}{\partial n} = T_B$ on $\partial\Omega$.

2.2.3 The dispersion model in the uniform reservoir

The distributed pollutant process satisfies the mass transfer equation, which includes transportation and diffusion. Averaging the equation over the depth, we get the convection-diffusion equation

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \tag{31}$$

where $C(x, y, t)$ is the concentration averaged in depth at the point (x, y) and at time t, D is the diffusion coefficient and $u(x, y, t), v(x, y, t)$ are the velocity component. We will consider the initial condition $C(x, y, 0) = g(x, y)$ at $t = 0$ and the boundary condition $\frac{\partial C}{\partial n} = 0$ on $\partial\Omega$.

CHAPTER 3

NUMERICAL TECHNIQUES

3.1 The numerical solution of hydrodynamic model

The hydrodynamic model provides the velocity field and elevation of the water. The calculated results of the model are input to the dispersion model, which provides the pollutant concentration field. To obtain an approximate solution of the initial boundary value problem, the finite difference methods is used.

3.1.1 The numerical solution for the hydrodynamic model in the uniform reservoir

The hydrodynamic model provides the velocity field and elevation of the water. The calculated results of the model are input to the dispersion model, which provides the pollutant concentration field. The equation (24)-(26) can be written in the form

$$\frac{\partial W}{\partial T} = A\frac{\partial W}{\partial X} + B\frac{\partial W}{\partial Y}, \qquad (32)$$

where,

$$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}, \quad A = \begin{bmatrix} 0 & -1 & 0 \\ -(1+XY) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+XY) & 0 & 0 \end{bmatrix}, \qquad (33)$$

$W_1 = Z, W_2 = U, W_3 = V$. We now discretize Eq. (32) by dividing the interval [0,1] into L and M subintervals and such that $L\Delta X = 1$ and $M\Delta Y = 1$, and the intervals $[0, \Im]$ into N subintervals such that $N\Delta T = \Im$. We can then approximate $W_1(X_l, Y_m, T_n)$ by $W^n_{1_{l,m}}$,the value of the difference approximation of $W_1(X,Y,T)$ at point $X = l\Delta X, Y = m\Delta Y$ and $T = n\Delta T$,where $0 \le l \le L, 0 \le m \le M$ and $0 \le n \le N$, and similarly defined for $W^n_{2_{l,m}}$ and $W^n_{3_{l,m}}$. The grid point (X_l, Y_m, T_n) are defined by $X_l = l\Delta X$ for all l = 0, 1, 2,…,L, $Y_m = m\Delta Y$ for all m = 0, 1, 2,…,M and $T_n = n\Delta T$ for all n = 0, 1, 2,…,N in which L, M and N are positive integers.

Using the Lax-Wendroff method (Mitchell, 1969). We have the following finite difference equation.

$$U_{l,m}^{n+1} = [I + \frac{1}{2}pA_{l,m}^{n+\frac{1}{2}}(\Delta_x + \nabla_x) + \frac{1}{2}pB_{l,m}^{n+\frac{1}{2}}(\Delta_y + \nabla_y) + \frac{1}{4}p^2(A_{l,m}^{n+\frac{1}{2}}\Delta_x A_{l,m}^{n+\frac{1}{2}}\nabla_x + A_{l,m}^{n+\frac{1}{2}}\nabla_x A_{l,m}^{n+\frac{1}{2}}\Delta_x)$$
$$+\frac{1}{4}p^2(B_{l,m}^{n+\frac{1}{2}}\Delta_y B_{l,m}^{n+\frac{1}{2}}\nabla_y + B_{l,m}^{n+\frac{1}{2}}\nabla_y B_{l,m}^{n+\frac{1}{2}}\Delta_y) + \frac{1}{8}p^2(A_{l,m}^{n+\frac{1}{2}}B_{l,m}^{n+\frac{1}{2}} + B_{l,m}^{n+\frac{1}{2}}A_{l,m}^{n+\frac{1}{2}})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y)]U_{l,m}^n, \tag{34}$$

where I is the unit matrix of order 3 and $p = \Delta t / \Delta x.$

Since A and B independent of time, we can assume that $A_{l,m}^n = A_{l,m}^{n+\frac{1}{2}} = A_{l,m}$ and $B_{l,m}^n = B_{l,m}^{n+\frac{1}{2}} = B_{l,m}$. It follows that $U_{l,m}^{n+\frac{1}{2}} = W_{l,m}^{n+\frac{1}{2}}$ and $U_{l,m}^n = W_{l,m}^n$.

where

$$W_{l,m}^n = \begin{pmatrix} W_{1_{l,m}}^n \\ W_{2_{l,m}}^n \\ W_{3_{l,m}}^n \end{pmatrix}, \nabla_x W_{l,m}^n = W_i^n - W_{i-1}^n \text{ and } \Delta_x W_i^n = W_{i+1}^n - W_i^n$$

we can see that

$$W_{l,m}^{n+1} = [W_{l,m}^n + \frac{1}{2}pA_{l,m}(\Delta_x + \nabla_x)W_{l,m}^n + \frac{1}{2}pB_{l,m}(\Delta_y + \nabla_y)W_{l,m}^n$$
$$+\frac{1}{4}p^2(A_{l,m}\Delta_x A_{l,m}\nabla_x + A_{l,m}\nabla_x A_{l,m}\Delta_x)W_{l,m}^n + \frac{1}{4}p^2(B_{l,m}\Delta_y B_{l,m}\nabla_y + B_{l,m}\nabla_y B_{l,m}\Delta_y)W_{l,m}^n$$
$$+\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y)W_{l,m}^n]. \tag{35}$$

Consider,
$$\frac{1}{2}pA_{l,m}(\Delta_x + \nabla_x)W_{l,m}^n = \frac{1}{2}pA_{l,m}(\Delta_x W_{l,m}^n + \nabla_x W_{l,m}^n),$$
$$= \frac{1}{2}pA_{l,m}(W_{l+1,m}^n - W_{l,m}^n + W_{l,m}^n - W_{l-1,m}^n),$$
$$= \frac{1}{2}pA_{l,m}(W_{l+1,m}^n - W_{l-1,m}^n). \tag{36}$$

Consider,
$$\frac{1}{2}pB_{l,m}(\Delta_y + \nabla_y)W_{l,m}^n = \frac{1}{2}pB_{l,m}(\Delta_y W_{l,m}^n + \nabla_y W_{l,m}^n),$$
$$\frac{1}{2}pB_{l,m}(\Delta_y + \nabla_y)W_{l,m}^n = \frac{1}{2}pB_{l,m}(W_{l,m+1}^n - W_{l,m}^n + W_{l,m}^n - W_{l,m-1}^n),$$
$$= \frac{1}{2}pB_{l,m}(W_{l,m+1}^n - W_{l,m-1}^n). \tag{37}$$

Consider, $$\frac{1}{4}p^2(A_{l,m}\Delta_x A_{l,m}\nabla_x + A_{l,m}\nabla_x A_{l,m}\Delta_x)W_{l,m}^n$$
$$=\frac{1}{4}p^2 A_{l,m}(\Delta_x A_{l,m}\nabla_x W_{l,m}^n + \nabla_x A_{l,m}\Delta_x W_{l,m}^n),$$
$$=\frac{1}{4}p^2 A_{l,m}(\Delta_x A_{l,m}(W_{l,m}^n - W_{l-1,m}^n) + \nabla_x A_{l,m}(W_{l+1,m}^n - W_{l,m}^n),$$
$$=\frac{1}{4}p^2 A_{l,m}(\Delta_x A_{l,m}W_{l,m}^n - \Delta_x A_{l,m}W_{l-1,m}^n) + (\nabla_x A_{l,m}W_{l+1,m}^n - \nabla_x A_{l,m}W_{l,m}^n), \qquad (38)$$

we obtain $$\frac{1}{4}p^2(A_{l,m}\Delta_x A_{l,m}\nabla_x + A_{l,m}\nabla_x A_{l,m}\Delta_x)W_{l,m}^n$$
$$=\frac{1}{4}p^2 A_{l,m}[(A_{l+1,m}W_{l+1,m}^n - A_{l,m}W_{l,m}^n) - (A_{l+1,m}W_{l,m}^n - A_{l,m}W_{l-1,m}^n)$$
$$+(A_{l,m}W_{l+1,m}^n - A_{l-1,m}W_{l,m}^n) - (A_{l,m}W_{l,m}^n - A_{l-1,m}W_{l-1,m}^n)]. \qquad (39)$$

Consider, $$\frac{1}{4}p^2(B_{l,m}\Delta_y B_{l,m}\nabla_y + B_{l,m}\nabla_y B_{l,m}\Delta_y)W_{l,m}^n$$
$$=\frac{1}{4}p^2 B_{l,m}(\Delta_y B_{l,m}\nabla_y + \nabla_y B_{l,m}\Delta_y)W_{l,m}^n,$$
$$=\frac{1}{4}p^2 B_{l,m}(\Delta_y B_{l,m}\nabla_y W_{l,m}^n + \nabla_y B_{l,m}\Delta_y W_{l,m}^n),$$
$$=\frac{1}{4}p^2 B_{l,m}(\Delta_y B_{l,m}(W_{l,m}^n - W_{l,m-1}^n) + \nabla_y B_{l,m}(W_{l,m+1}^n - W_{l,m}^n),$$
$$=\frac{1}{4}p^2 B_{l,m}(\Delta_y B_{l,m}W_{l,m}^n - \Delta_y B_{l,m}W_{l,m-1}^n) + (\nabla_y B_{l,m}W_{l,m+1}^n - \nabla_y B_{l,m}W_{l,m}^n), \qquad (40)$$

we obtain $$\frac{1}{4}p^2(B_{l,m}\Delta_y B_{l,m}\nabla_y + B_{l,m}\nabla_y B_{l,m}\Delta_y)W_{l,m}^n$$
$$=\frac{1}{4}p^2 B_{l,m}[(B_{l,m+1}W_{l,m+1}^n - B_{l,m}W_{l,m}^n) - (B_{l,m+1}W_{l,m}^n - B_{l,m}W_{l,m-1}^n)$$
$$+(B_{l,m}W_{l,m+1}^n - B_{l,m-1}W_{l,m}^n) - (B_{l,m}W_{l,m}^n - B_{l,m-1}W_{l,m-1}^n)]. \qquad (41)$$

Consider, $$\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y)W_{l,m}^n$$
$$=\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(\Delta_x + \nabla_x)(\Delta_y W_{l,m}^n + \nabla_y W_{l,m}^n),$$
$$=\frac{1}{8}p^2(A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(\Delta_x + \nabla_x)(W_{l,m+1}^n - W_{l,m}^n + W_{l,m}^n - W_{l,m-1}^n),$$

$$
\begin{aligned}
&= \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})(\Delta_x + \nabla_x)(W_{l,m+1}^n - W_{l,m-1}^n), \\
&= \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})[(\Delta_x + \nabla_x) W_{l,m+1}^n - (\Delta_x + \nabla_x) W_{l,m-1}^n], \\
&= \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})[(\Delta_x W_{l,m+1}^n + \nabla_x W_{l,m+1}^n) - (\Delta_x W_{l,m-1}^n + \nabla_x W_{l,m-1}^n)], \\
&= \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})[(W_{l+1,m+1}^n - W_{l,m+1}^n) + (W_{l,m+1}^n - W_{l-1,m+1}^n)] \\
&\quad -[(W_{l+1,m-1}^n - W_{l,m-1}^n) + (W_{l,m-1}^n - W_{l-1,m-1}^n)], \qquad (42)
\end{aligned}
$$

we obtain

$$
\frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})(\Delta_x + \nabla_x)(\Delta_y + \nabla_y) W_{l,m}^n = \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})[(W_{l+1,m+1}^n - W_{l-1,m+1}^n - W_{l+1,m-1}^n + W_{l-1,m-1}^n)]. \qquad (43)
$$

It obtained the general form of equation,

$$
\begin{aligned}
W_{l,m}^{n+1} = W_{l,m}^n &+ \frac{1}{2} p A_{l,m} (W_{l+1,m}^m - W_{l-1,m}^n) + \frac{1}{2} p B_{l,m} (W_{l,m+1}^n - W_{l,m-1}^n) \\
&+ \frac{1}{4} p^2 A_{l,m} [(A_{l+1,m} W_{l+1,m}^n - A_{l,m} W_{l,m}^n) - (A_{l+1,m} W_{l,m}^n - A_{l,m} W_{l-1,m}^n) \\
&+ (A_{l,m} W_{l+1,m}^n - A_{l-1,m} W_{l,m}^n) - (A_{l,m} W_{l,m}^n - A_{l-1,m} W_{l-1,m}^n)] \\
&+ \frac{1}{4} p^2 B_{l,m} [(B_{l,m+1} W_{l,m+1}^n - B_{l,m} W_{l,m}^n) - (B_{l,m+1} W_{l,m}^n - B_{l,m} W_{l,m-1}^n) \\
&+ (B_{l,m} W_{l,m+1}^n - B_{l,m-1} W_{l,m}^n) - (B_{l,m} W_{l,m}^n - B_{l,m-1} W_{l,m-1}^n)] \\
&+ \frac{1}{8} p^2 (A_{l,m} B_{l,m} + B_{l,m} A_{l,m})(W_{l+1,m+1}^n - W_{l-1,m+1}^n - W_{l+1,m-1}^n + W_{l-1,m-1}^n), \qquad (44)
\end{aligned}
$$

where

$$
W_{l,m}^n = \begin{pmatrix} W_{1_{l,m}}^n \\ W_{2_{l,m}}^n \\ W_{3_{l,m}}^n \end{pmatrix}, \qquad \Delta_x W_i^n = W_{i+1}^n - W_i^n, \qquad \nabla_x W_i^n = W_i^n - W_{i-1}^n.
$$

A stability analysis of the Lax-Wendroff scheme (44) with matrices $A_{l,m}$ and $B_{l,m}$ has been given in (Mitchell, 1969). The Lax-Wendroff scheme is stable if $p|\lambda_0| \le \frac{1}{2\sqrt{2}}$, where

$|\lambda_0| = \max\{|\lambda_{A_{l,m}}|, |\lambda_{B_{l,m}}|\}$ such that $|A - \lambda_A I| = 0, |B - \lambda_B I| = 0,$ where $\lambda_{A_{l,m}}, \lambda_{B_{l,m}}$ are eigenvalues of $A_{l,m}$ and $B_{l,m}$ respectively. We can obtain in our case the $\lambda_A = 1, -1, 0$ and $\lambda_B = -1, 0, 1$, it follows that $|\lambda_0| = 1$. Then p must be less than or equal to $\dfrac{1}{2\sqrt{2}}$.

3.2 The numerical method for dispersion model in the uniform reservoir

We can then approximate $C(x_l, y_m, t_n)$ by $C_{l,m}^n$, the value of the difference approximation of $C(x, y, t)$ at point $x = l\Delta x$, $y = m\Delta y$ and $t = n\Delta i$, where $0 \le l \le L,$ $0 \le m \le M$ and $0 \le n \le N.$ The grid point (x_l, y_m, t_n) are defined by $x_l = l\Delta x$ for all $l = 0,1,2,\ldots,L,$ $y_m = m\Delta y$ for all $m = 0,1,2,\ldots,M,$ and $t_n = n\Delta i$ for all $n = 0,1,2,\ldots,N$ in which L, M and N are positive integers. Consider forward in time technique

$$\frac{\partial U}{\partial t} \approx \frac{U(x, t + \Delta t) - U(x, t)}{\Delta t}, \tag{45}$$

Consider the Taylor series expanise

$$U(x + \Delta x) = U(x) + \Delta x U'(x) + \frac{(\Delta x)^2}{2!} U''(x) + \frac{(\Delta x)^3}{3!} U'''(x) + \ldots, \tag{46}$$

$$U(x - \Delta x) = U(x) - \Delta x U'(x) + \frac{(\Delta x)^2}{2!} U''(x) - \frac{(\Delta x)^3}{3!} U'''(x) + \ldots, \tag{47}$$

If Eq(46) and Eq(47) are subtracted, the term involving $U''(x)$ and $U^{(4)}(x)$ are eliminated and simple algebraic maniputation gives

$$U(x + \Delta x) - U(x - \Delta x) = 2\Delta x U'(x) + \frac{(\Delta x)^3}{3!} U'''(x) + \ldots, \tag{48}$$

$$U(x + \Delta x) - U(x - \Delta x) = 2\Delta x U'(x) + O\left(\left(\Delta x\right)^2\right), \tag{49}$$

hence,

$$U'(x) \approx \frac{U(x + \Delta x) - U(x - \Delta x)}{2\Delta x}, \tag{50}$$

where O(Δx) denote term containing Δx and higher power of Δx. If Eq(46) and Eq(47) are added, we can see that

$$U(x + \Delta x) + U(x - \Delta x) = 2U(x) + (\Delta x)^2 U''(x) + \frac{(\Delta x)^4}{4!} U^{(4)}(x) + \ldots, \tag{51}$$

$$U(x+\Delta x)+U(x-\Delta x)=2U(x)+(\Delta x)^2U''(x)+O\left((\Delta x)^2\right), \tag{52}$$

hence,

$$U''(x)\approx\frac{U(x+\Delta x)-2U(x)+U(x-\Delta x)}{(\Delta x)^2}. \tag{53}$$

Taking the forward difference in time and central difference in space technigue in equation (31) consider

$$\frac{C_{l,m}^{n+1}-C_{l,m}^{n}}{\Delta t}+u_{l.m}^{n}\left(\frac{C_{l+1,m}^{n}-C_{l-1,m}^{n}}{2\Delta x}\right)+v_{l,m}^{n}\left(\frac{C_{l,m+1}^{n}-C_{l,m-1}^{n}}{2\Delta y}\right)$$
$$=D\left(\frac{C_{l+1,m}^{n}-2C_{l,m}^{n}+C_{l-1,m}^{n}}{(\Delta x)^2}+\frac{C_{l,m+1}^{n}-2C_{l,m}^{n}+C_{l,m-1}^{n}}{(\Delta y)^2}\right), \tag{54}$$

multiply Eq(54) by Δt, we can then

$$C_{l,m}^{n+1}-C_{l,m}^{n}+\frac{\Delta t}{2\Delta x}u_{l.m}^{n}\left(C_{l+1,m}^{n}-C_{l-1,m}^{n}\right)+\frac{\Delta t}{2\Delta y}v_{l,m}^{n}\left(C_{l,m+1}^{n}-C_{l,m-1}^{n}\right)$$
$$=D\left[\frac{\Delta t}{(\Delta x)^2}(C_{l+1,m}^{n}-2C_{l,m}^{n}+C_{l-1,m}^{n})+\frac{\Delta t}{(\Delta y)^2}(C_{l,m+1}^{n}-2C_{l,m}^{n}+C_{l,m-1}^{n})\right], \tag{55}$$

we get

$$C_{l,m}^{n+1}-C_{l,m}^{n}+\frac{\Delta t}{2\Delta x}u_{l.m}^{n}C_{l+1,m}^{n}-\frac{\Delta t}{2\Delta x}u_{l.m}^{n}C_{l-1,m}^{n}+\frac{\Delta t}{2\Delta y}v_{l,m}^{n}C_{l,m+1}^{n}-\frac{\Delta t}{2\Delta y}v_{l,m}^{n}C_{l,m-1}^{n}=\frac{\Delta t}{(\Delta x)^2}DC_{l+1,m}^{n}$$

$$-\frac{\Delta t}{(\Delta x)^2}2DC_{l,m}^{n}+\frac{\Delta t}{(\Delta x)^2}DC_{l-1,m}^{n}+\frac{\Delta t}{(\Delta y)^2}D(C_{l,m+1}^{n}-\frac{\Delta t}{(\Delta y)^2}2DC_{l,m}^{n}+\frac{\Delta t}{(\Delta y)^2}DC_{l,m-1}^{n}, \tag{56}$$

we can see that

$$C_{l,m}^{n+1}=C_{l,m}^{n}-\frac{\Delta t}{2\Delta x}u_{l.m}^{n}C_{l+1,m}^{n}+\frac{\Delta t}{2\Delta x}u_{l.m}^{n}C_{l-1,m}^{n}-\frac{\Delta t}{2\Delta y}v_{l,m}^{n}C_{l,m+1}^{n}+\frac{\Delta t}{2\Delta y}v_{l,m}^{n}C_{l,m-1}^{n}+\frac{\Delta t}{(\Delta x)^2}DC_{l+1,m}^{n}$$
$$-\frac{\Delta t}{(\Delta x)^2}2DC_{l,m}^{n}+\frac{\Delta t}{(\Delta x)^2}DC_{l-1,m}^{n}+\frac{\Delta t}{(\Delta y)^2}DC_{l,m+1}^{n}-\frac{\Delta t}{(\Delta y)^2}2DC_{l,m}^{n}+\frac{\Delta t}{(\Delta y)^2}DC_{l,m-1}^{n}, \tag{57}$$

we have

$$C_{l,m}^{n+1} = C_{l,m}^{n} - \frac{\Delta t}{(\Delta x)^2} 2DC_{l,m}^{n} - \frac{\Delta t}{(\Delta y)^2} 2DC_{l,m}^{n} - \frac{\Delta t}{2\Delta y} v_{l,m}^{n} C_{l,m+1}^{n} + \frac{\Delta t}{(\Delta y)^2} DC_{l,m+1}^{n} + \frac{\Delta t}{(\Delta x)^2} DC_{l+1,m}^{n}$$

$$- \frac{\Delta t}{2\Delta x} u_{l.m}^{n} C_{l+1,m}^{n} + \frac{\Delta t}{2\Delta x} u_{l.m}^{n} C_{l-1,m}^{n} + \frac{\Delta t}{(\Delta x)^2} DC_{l-1,m}^{n} + \frac{\Delta t}{(\Delta y)^2} DC_{l,m-1}^{n} + \frac{\Delta t}{2\Delta y} v_{l,m}^{n} C_{l,m-1}^{n}, \qquad (58)$$

we can see that

$$C_{l,m}^{n+1} = \left(\frac{\Delta t}{(\Delta x)^2} D + \frac{\Delta t}{2\Delta x} u_{l.m}^{n} \right) C_{l-1,m}^{n} \left(1 - \frac{\Delta t}{(\Delta x)^2} 2D - \frac{\Delta t}{(\Delta y)^2} 2D \right) C_{l,m}^{n} + \left(\frac{\Delta t}{(\Delta x)^2} D - \frac{\Delta t}{2\Delta x} u_{l.m}^{n} \right) C_{l+1,m}^{n}$$

$$+ \left(\frac{\Delta t}{(\Delta y)^2} D + \frac{\Delta t}{2\Delta y} v_{l,m}^{n} \right) C_{l,m-1}^{n} + \left(\frac{\Delta t}{(\Delta y)^2} D - \frac{\Delta t}{2\Delta y} v_{l,m}^{n} \right) C_{l,m+1}^{n}, \qquad (59)$$

for $1 \le l \le L-1,\ 1 \le m \le M-1$ and $0 \le n \le N-1.$ Let $\lambda = \frac{\Delta t}{(\Delta x)^2} = \frac{\Delta t}{(\Delta y)^2}$, Eq(59) becomes

$$C_{l,m}^{n+1} = \left(\lambda D + \frac{\Delta t}{2\Delta x} u_{l.m}^{n} \right) C_{l-1,m}^{n} \left(1 - 2\lambda D - 2\lambda D \right) C_{l,m}^{n} + \left(\lambda D - \frac{\Delta t}{2\Delta x} u_{l.m}^{n} \right) C_{l+1,m}^{n}$$
$$+ \left(\lambda D + \frac{\Delta t}{2\Delta y} v_{l,m}^{n} \right) C_{l,m-1}^{n} + \left(\lambda D - \frac{\Delta t}{2\Delta y} v_{l,m}^{n} \right) C_{l,m+1}^{n}. \qquad (60)$$

CHAPTER 4

NUMERICAL EXAMPLES

We consider a uniform reservoir with dimension 320x320 m $(l = 320m)$ and constant depth $h = 1m.$ The reservoir is meshed by 25 grids points with $\Delta x = \Delta y = 80m$ and we take the time interval $\Delta t = 5s.$ Initially the water in the reservoir is assumed to be motionless $(u = 0,$ $v = 0)$ and the water elevation is specified as $\zeta(x, y, 0) = \left(\frac{x}{l} - \frac{x^2}{l^2}\right)\left(\frac{y}{l} - \frac{y^2}{l^2}\right).$ The pollutant concentration in the reservoir is $C(x, y, 0) = x^2\left(1 - \frac{x^2}{2}\right)y^2\left(1 - \frac{y^2}{2}\right)$ with $\frac{\partial C}{\partial n} = 0,$ non-absorbing boundary of the reservoir. Using Eq. (44), the water elevation and the velocities in x-direction and y-direction are shown in Table 1-9 respectively (See Appendix A).

Table 1 The elevation ζ (dimensionless) at T=0

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.035156	0.046875	0.035156	0
0.50	0	0.046875	0.062500	0.046875	0
0.75	0	0.035156	0.046875	0.035156	0
1	0	0	0	0	0

Table 2 The elevation ζ (dimensionless) at T=0.05

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.03420	0.04590	0.03490	0
0.50	0	0.04580	0.06130	0.04540	0
0.75	0	0.03410	0.04580	0.03350	0
1	0	0	0	0	0

Table 3 The elevation ζ (dimensionless) at T=0.1

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.03324	0.04371	0.03387	0
0.50	0	0.04360	0.05686	0.04301	0
0.75	0	0.03315	0.04335	0.03259	0
1	0	0	0	0	0

In table 1-3 the water elevation at center point of the reservoir is greater than another point. At the same point in the next time step, the water elevation will going down.

Table 4 The velocity x-direction u (dimensionless) T = 0

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0	0	0	0
0.50	0	0	0	0	0
0.75	0	0	0	0	0
1	0	0	0	0	0

Table 5 The velocity x-direction u (dimensionless) T = 0.05

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	-0.01565	-0.01565	0	0.01753	0.01753
0.50	-0.02191	-0.02191	0	0.02692	0.02692
0.75	-0.01784	-0.01784	0	0.02285	0.02285
1	0	0	0	0	0

Table 6 The velocity x-direction u (dimensionless) T = 0.1

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	-0.03024	-0.03024	-0.00019	0.03206	0.03206
0.50	-0.04110	-0.04110	0.00028	0.04589	0.04589
0.75	-0.03221	-0.03221	0.00031	0.03713	0.03713
1	0	0	0	0	0

In table 4-6, the velocity of water current in x-direction, the speed has change from the originally.

Table 7 The velocity y-direction v (dimensionless) T = 0

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0	0	0	0
0.50	0	0	0	0	0
0.75	0	0	0	0	0
1	0	0	0	0	0

Table 8 The velocity y-direction v (dimensionless) T = 0.05

y , x	0	0.25	0.50	0.75	1
0	0	-0.01565	-0.02191	-0.01753	0
0.25	0	-0.01565	-0.02191	-0.01753	0
0.50	0	0	0	0	0
0.75	0	0.01784	0.02692	0.02285	0
1	0	0.01784	0.02692	0.02285	0

hk

Table 9 The velocity y-direction v (dimensionless) T = 0.1

y , x	0	0.25	0.50	0.75	1
0	0	-0.03021	-0.04110	-0.03190	0
0.25	0	-0.03021	-0.04110	-0.03190	0
0.50	0	0.00006	0.00019	0.00056	0
0.75	0	0.03221	0.04589	0.03697	0
1	0	0.03221	0.04589	0.03697	0

In table 7-9 at the same point when the time increase the velocity of water in the y-direction, the speed increase in the direction originally.

Table 10 The initial pollutant concentration $C\left(x,y,t\right)kg/m^3$ for $0 \le x, y \le 1$at t = 0

y , x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.003666	0.013245	0.024479	0
0.50	0	0.013245	0.047852	0.088440	0
0.75	0	0.024479	0.088440	0.163456	0
1	0	0	0	0	0

Applying equation (60) with the values of u and v in Tables 5-6 and 8-9, we get the pollutant concentration represented in Table 11-12 respectively (See Appendix B).

Table 11 The pollutant concentration $C\left(x,y,t\right)kg/m^3$ for $0 \le x, y \le 1$ at t = 0.05

y , x	0	0.25	0.50	0.75	1
0	0.000881	0.000881	0.003208	0.005918	0.005918
0.25	0.000886	0.006546	0.018874	0.025562	0.005918
0.50	0.003208	0.018874	0.050723	0.060255	0.021464
0.75	0.005919	0.025565	0.060255	0.049394	0.039603
1	0.005919	0.005919	0.021464	0.039603	0.039603

Table 12 The pollutant concentration $C(x,y,t)\,kg/m^3$ for $0 \le x, y \le 1$ at t = 0.1

y , x	0	0.25	0.50	0.75	1
0	0.000887	0.000887	0.003233	0.005953	0.005953
0.25	0.000891	0.006548	0.018966	0.025708	0.005953
0.50	0.003233	0.018965	0.050719	0.060338	0.021631
0.75	0.005954	0.025711	0.060342	0.049645	0.039836
1	0.005954	0.005954	0.021631	0.039834	0.039834

In table 10-12, the pollutant concentration level will be growing up.

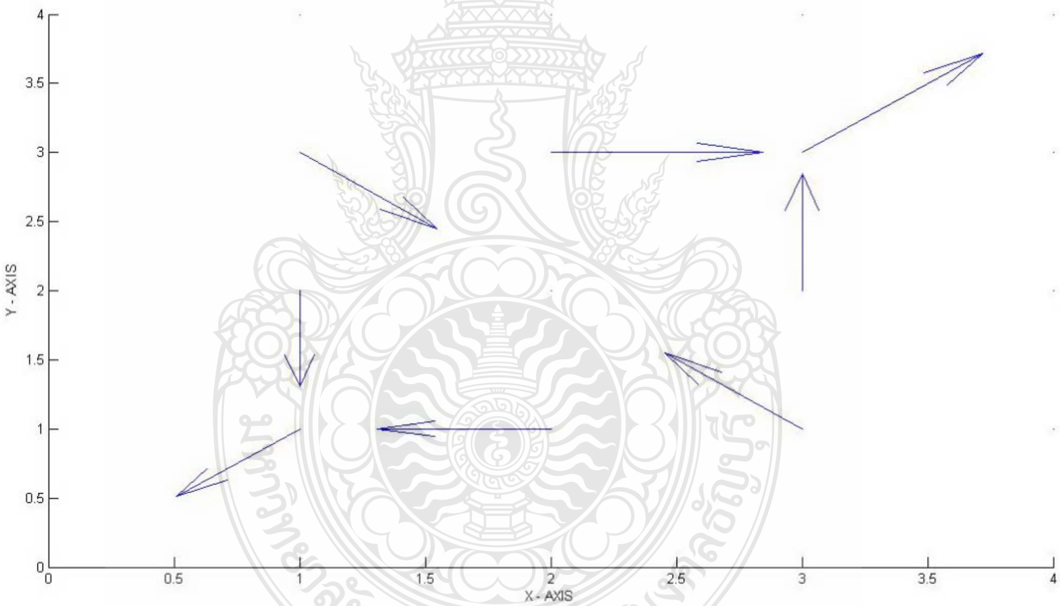


Figure 3 The direction result of velocity u and v

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

A numerical solutions of hydrodynamic model that provides the velocity and elevation of water in the reservoir is presented. The numerical techniques can transform the original hydrodynamic model to be non-dimensional form of equation. The numerical results can also transform come back to be dimensional solution as the input data for the convection-diffusion equation of the water quality model.

These models can be applied to the real cases for sewage effluent in the non-uniform reservoir, which changing the inputs water elevation function and discharge pollutant concentration function.

List of Bibliography

Mitchell, A.R. **Computational Methods in Partial Differential Equation**. New York: Wiley, 1969.

Ninomiya, H. and Onishi, K. **Flow Analysis Using a PC** : Computational Mechanics Publications. Avon: CRC Press, 1991.

Pochai, N., Tangmanee, S., Crane, LJ. and Miller, J.J.M. "A mathematical model of water pollution control using the finite element method," **Proceedings in Applied Mathematics and Mechanics.** 6, 1(January 2006): 755-756.

Pochai, N., Tangmanee, S., Crane, LJ. and Miller, J.J.H. "A water quality computation in the uniform reservoir," **Journal of Interdisciplinary Mathematics**. 11, 6(June 2008): 803-814.

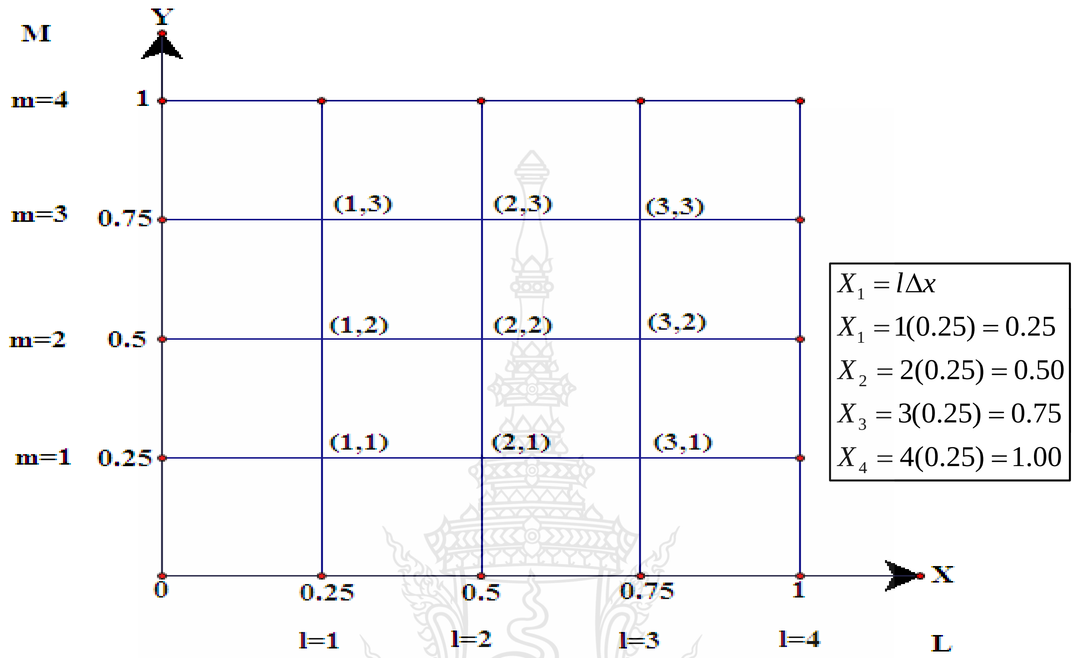
Pochai, N. "A Numerical Computation of the Non-Dimensional Form of a Non-Linear Hydrodynamic Model in a Uniform Reservoir," **Nonlinear Analysis : Hybrid Systems.** 3, (March 2009): 463-466.

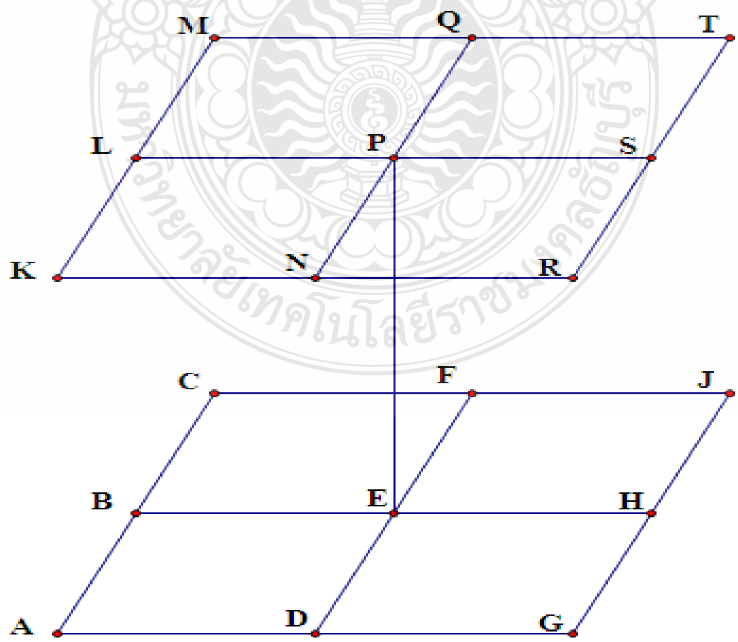
Pochai, N. and Chunya, S. "A Non-Dimensional Form of Hydrodynamic Model with Variable Coefficients in a Uniform Reservoir Using Lax-Wendroff Method," **Physics Procedia**. 8, (August 2011): 89-93.

Smith, G.D. **Numerical Solution of Partial Differential Equations**. Oxford: Billing and Sons Limited Guildford, 1978.

Tabuenca, P., Vila, J., Cardona, J. and Samartin, A. "Finite element simulation of dispersion in the bay of Santander," **Advanced in Engineering Software.** 28, (April 1997): 313-332.

A numerical manipulation of Lax-Wendroff method for hydrodynamic model.





Lax (1, 0) Initial Step

$$Z = (1-X)X(1-Y)Y = f(x,y)$$

At $x = 0.25, \quad y = 0.25$

Then $Z_{1,1} = (1-0.25)(0.25)(1-0.25)(0.25) = 0.03516$

z=(1-X)X(1-Y)Y

	X	Y	Z
$Z_{1,1}$	0.25	0.25	0.035156
$Z_{1,2}$	0.25	0.5	0.046875
$Z_{1,3}$	0.25	0.75	0.035156
$Z_{2,1}$	0.5	0.25	0.046875
$Z_{2,2}$	0.5	0.5	0.062500
$Z_{2,3}$	0.5	0.75	0.046875
$Z_{3,1}$	0.75	0.25	0.035156
$Z_{3,2}$	0.75	0.5	0.046875
$Z_{3,3}$	0.75	0.75	0.035156

Lax (2, 0.05)

$\Delta t = 0.05$

	$W^1_{1,1}$	$W^1_{1,2}$	$W^1_{1,3}$	$W^1_{2,1}$	$W^1_{2,2}$	$W^1_{2,3}$	$W^1_{3,1}$	$W^1_{3,2}$	$W^1_{3,3}$
Z	0.03420	0.04580	0.03410	0.04590	0.06130	0.04580	0.03490	0.04540	0.03350
ζ	0.03420	0.04580	0.03410	0.04590	0.06130	0.04580	0.03490	0.04540	0.03350
U	-0.00500	-0.00700	-0.00570	0.00000	0.00000	0.00000	0.00560	0.00860	0.00730
u	-0.01565	-0.02191	-0.01784	0.00000	0.00000	0.00000	0.01753	0.02692	0.02285
V	-0.00500	0.00000	0.00570	-0.00700	0.00000	0.00860	-0.00560	0.00000	0.00730
v	-0.01565	0.00000	0.01784	-0.02191	0.00000	0.02692	-0.01753	0.00000	0.02285

$Z = \frac{\zeta}{h}$ $\zeta = Zh$ $U = \frac{u}{\sqrt{gh}}$ $u = U\sqrt{gh}$ $u = U\sqrt{(9.8)(1)}$

$V = \frac{v}{\sqrt{gh}}$ $v = V\sqrt{gh}$ $v = V\sqrt{(9.8)(1)}$

	$W^1_{1,3}$	$W^1_{2,3}$	$W^1_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

ζ

0	0	0	0	0
0	0.03410	0.04580	0.03350	0
0	0.04580	0.06130	0.04540	0
0	0.03420	0.04590	0.03490	0
0	0	0	0	0

	$W_{1,3}$	$W_{2,3}$	$W_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

u

0	0	0	0	0
-0.01784	-0.01784	0.00000	0.02285	0.02285
-0.02191	-0.02191	0.00000	0.02692	0.02692
-0.01565	-0.01565	0.00000	0.01753	0.01753
0	0	0	0	0

	$W_{1,3}$	$W_{2,3}$	$W_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

v

0	0.01784	0.02692	0.02285	0
0	0.01784	0.02692	0.02285	0
0	0.00000	0.00000	0.00000	0
0	-0.01565	-0.02191	-0.01753	0
0	-0.01565	-0.02191	-0.01753	0

Lax (2, 0.05)

$$\Delta x = \Delta y = 0.25 \, , \Delta t = 0.05$$

$$p = \frac{\Delta t}{\Delta x} = \frac{0.05}{0.25} = 0.2$$

$$U = 0, V = 0$$

$$W_{l,m}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{l,m}^{0} = \begin{pmatrix} X(1-X)Y(1-Y) \\ 0 \\ 0 \end{pmatrix}$$

$$W_{l,m}^{n+1} = W_{l,m}^{n} + \frac{1}{2} pA_{l,m}(W_{l+1,m}^{n} - W_{l-1,m}^{n}) + \frac{1}{2} pB_{l,m}(W_{l,m+1}^{n} - W_{l,m-1}^{n})$$

$$+\frac{1}{4} p^{2} A_{l,m}[(A_{l+1,m}W_{l+1,m}^{n} - A_{l,m}W_{l,m}^{n}) - (A_{l+1,m}W_{l,m}^{n} - A_{l,m}W_{l-1,m}^{n}) +$$

$$(A_{l,m}W_{l+1,m}^{n} - A_{l-1,m}W_{l,m}^{n}) - (A_{l,m}W_{l,m}^{n} - A_{l-1,m}W_{l-1,m}^{n})] +$$

$$+\frac{1}{4} p^{2} B_{l,m}[(B_{l,m+1}W_{l,m+1}^{n} - B_{l,m}W_{l,m}^{n}) - (B_{l,m+1}W_{l,m}^{n} - B_{l,m}W_{l,m-1}^{n}) +$$

$$(B_{l,m}W_{l,m+1}^{n} - B_{l,m-1}W_{l,m}^{n}) - (B_{l,m}W_{l,m}^{n} - B_{l,m-1}W_{l,m-1}^{n})]$$

$$+\frac{1}{8} p^{2}[A_{l,m}B_{l,m} + B_{l,m}A_{l,m})(W_{l+1,m+1}^{n} - W_{l+1,m-1}^{n} - W_{l-1,m+1}^{n} + W_{l-1,m-1}^{n})$$

$$l = 1, \quad m = 1, \quad X = l\Delta x, \quad Y = m\Delta y$$

$$A_{1,1} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+XY) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+lm(\Delta x)^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+(1)(1)(0.25)^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{1,1} = \begin{bmatrix} 0 & -1 & 0 \\ -1.0625 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

	l	Δ_x	X=LΔX	m	Δy	Y=mΔY	-(1+XY)
$A_{1,1}$	1	0.25	0.25	1	0.25	0.25	-1.0625
$A_{0,1}$	0	0.25	0	1	0.25	0.25	-1.0000
$A_{0,2}$	0	0.25	0	2	0.25	0.5	-1.0000
$A_{0,3}$	0	0.25	0	3	0.25	0.75	-1.0000
$A_{0,4}$	0	0.25	0	4	0.25	1	-1.0000
$A_{1,2}$	1	0.25	0.25	2	0.25	0.5	-1.1250
$A_{1,3}$	1	0.25	0.25	3	0.25	0.75	-1.1875
$A_{1,4}$	1	0.25	0.25	4	0.25	1	-1.2500
$A_{2,1}$	2	0.25	0.5	1	0.25	0.25	-1.1250
$A_{2,2}$	2	0.25	0.5	2	0.25	0.5	-1.2500
$A_{2,3}$	2	0.25	0.5	3	0.25	0.75	-1.3750
$A_{2,4}$	2	0.25	0.5	4	0.25	1	-1.5000
$A_{3,1}$	3	0.25	0.75	1	0.25	0.25	-1.1875
$A_{3,2}$	3	0.25	0.75	2	0.25	0.5	-1.3750
$A_{3,3}$	3	0.25	0.75	3	0.25	0.75	-1.5625
$A_{3,4}$	3	0.25	0.75	4	0.25	1	-1.7500
$A_{4,1}$	4	0.25	1	1	0.25	0.25	-1.2500
$A_{4,2}$	4	0.25	1	2	0.25	0.5	-1.5000
$A_{4,3}$	4	0.25	1	3	0.25	0.75	-1.7500
$A_{4,4}$	4	0.25	1	4	0.25	1	-2.0000

$l = 1, \quad m = 1, \quad X = l\Delta x, \quad Y = m\Delta y$

$$B_{1,1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+XY) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+lm(\Delta x)^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+(1)(1)(0.25) & 0 & 0 \end{bmatrix}$$

$$B_{1,1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.0625 & 0 & 0 \end{bmatrix}$$

	l	Δ_x	X=LΔX	m	Δy	Y=mΔY	-(1+XY)
$B_{1,1}$	1	0.25	0.25	1	0.25	0.25	-1.0625
$B_{0,1}$	0	0.25	0	1	0.25	0.25	-1.0000
$B_{0,2}$	0	0.25	0	2	0.25	0.5	-1.0000
$B_{0,3}$	0	0.25	0	3	0.25	0.75	-1.0000
$B_{0,4}$	0	0.25	0	4	0.25	1	-1.0000
$B_{1,2}$	1	0.25	0.25	2	0.25	0.5	-1.1250
$B_{1,3}$	1	0.25	0.25	3	0.25	0.75	-1.1875
$B_{1,4}$	1	0.25	0.25	4	0.25	1	-1.2500
$B_{2,1}$	2	0.25	0.5	1	0.25	0.25	-1.1250
$B_{2,2}$	2	0.25	0.5	2	0.25	0.5	-1.2500
$B_{2,3}$	2	0.25	0.5	3	0.25	0.75	-1.3750
$B_{2,4}$	2	0.25	0.5	4	0.25	1	-1.5000
$B_{3,1}$	3	0.25	0.75	1	0.25	0.25	-1.1875
$B_{3,2}$	3	0.25	0.75	2	0.25	0.5	-1.3750
$B_{3,3}$	3	0.25	0.75	3	0.25	0.75	-1.5625
$B_{3,4}$	3	0.25	0.75	4	0.25	1	-1.7500
$B_{4,1}$	4	0.25	1	1	0.25	0.25	-1.2500
$B_{4,2}$	4	0.25	1	2	0.25	0.5	-1.5000
$B_{4,3}$	4	0.25	1	3	0.25	0.75	-1.7500
$B_{4,4}$	4	0.25	1	4	0.25	1	-2.0000

$$Let \quad n=0, \quad l=1, \quad m=1$$

Then we obtain $\quad W_{l,m}^{n} = \begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix}$, $\quad W_{l,m}^{n} = W_{1,1}^{0} = \begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} = \begin{pmatrix} 0.03516 \\ 0 \\ 0 \end{pmatrix}$

$$W_{1,1}^{0} = \begin{pmatrix} Z_{1.1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} + \frac{1}{2}(0.2)A_{1,1}\left[\begin{pmatrix} Z_{2.1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix} - \begin{pmatrix} Z_{0.1} \\ U_{0,1} \\ V_{0,1} \end{pmatrix}\right] + \frac{1}{2}(0.2)B_{1,1}\left[\begin{pmatrix} Z_{1,2} \\ U_{1,2} \\ V_{1,2} \end{pmatrix} - \begin{pmatrix} Z_{1,0} \\ U_{1,0} \\ V_{1,0} \end{pmatrix}\right]$$

$$+\frac{1}{4}(0.2)^{2}A_{1,1}\left[A_{2,1}\begin{pmatrix} Z_{2.1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix} - A_{1,1}\begin{pmatrix} Z_{1.1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix}\right] - \left[A_{2,1}\begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} - A_{1,1}\begin{pmatrix} Z_{0,1} \\ U_{0,1} \\ V_{0,1} \end{pmatrix}\right]$$

$$+\left[A_{1,1}\begin{pmatrix} Z_{2.1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix} - A_{0,1}\begin{pmatrix} Z_{1.1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix}\right] - \left[A_{1,1}\begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} - A_{0,1}\begin{pmatrix} Z_{0,1} \\ U_{0,1} \\ V_{0,1} \end{pmatrix}\right]$$

$$+\frac{1}{4}(0.2)^{2}B_{1,1}\left[B_{1,2}\begin{pmatrix} Z_{1,2} \\ U_{1,2} \\ V_{1,2} \end{pmatrix} - B_{1,1}\begin{pmatrix} Z_{1.1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix}\right] - \left[B_{1,2}\begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} - B_{1,1}\begin{pmatrix} Z_{1,0} \\ U_{1,0} \\ V_{1,0} \end{pmatrix}\right]$$

$$+\left[B_{1,1}\begin{pmatrix} Z_{1,2} \\ U_{1,2} \\ V_{1,2} \end{pmatrix} - B_{1,0}\begin{pmatrix} Z_{1.1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix}\right] - \left[B_{1,1}\begin{pmatrix} Z_{1,1} \\ U_{1,1} \\ V_{1,1} \end{pmatrix} - B_{1,0}\begin{pmatrix} Z_{1,0} \\ U_{1,0} \\ V_{1,0} \end{pmatrix}\right]$$

$$+\frac{1}{8}(0.2)^{2}\left[A_{1,1}B_{1,1} + B_{1,1}A_{1,1}\right]\left[\begin{pmatrix} Z_{2,2} \\ U_{2,2} \\ V_{2,2} \end{pmatrix} - \begin{pmatrix} Z_{2.0} \\ U_{2.0} \\ V_{2,0} \end{pmatrix} - \begin{pmatrix} Z_{0,2} \\ U_{0,2} \\ V_{0,2} \end{pmatrix} - \begin{pmatrix} Z_{0,0} \\ U_{0,0} \\ V_{0,0} \end{pmatrix}\right]$$

$$
W_{1,1}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]
$$

$$
+0.01\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]
$$

$$
+\left[\left[\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]
$$

$$
+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]
$$

$$
+\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\begin{pmatrix}0.\\0\\0\end{pmatrix}\right]\right]
$$

$$
+0.005\left[\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.0625&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]
$$

We got,

$$
W_{1,1}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-0.0050\\0\end{pmatrix}+\begin{pmatrix}0\\0\\-0.0050\end{pmatrix}+\begin{pmatrix}-0.0005\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0005\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}
$$

$$
W_{1,1}^{0}=\begin{pmatrix}0.0342\\-0.0050\\-0.0050\end{pmatrix}
$$

then,

$$
\begin{pmatrix}Z_{1,1}\\U_{1,1}\\V_{1,1}\end{pmatrix}=\begin{pmatrix}0.0342\\-0.0050\\-0.0050\end{pmatrix}
$$

$$Let \quad n=0, \quad l=1, \quad m=2$$

Then we obtain $\quad W_{l,m}^{n} = \begin{pmatrix} Z_{1,2} \\ U_{1,2} \\ V_{1,2} \end{pmatrix}$, $\quad W_{l,m}^{n} = W_{1,2}^{0} = \begin{pmatrix} Z_{1,2} \\ U_{1,2} \\ V_{1,2} \end{pmatrix} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix}$

$$W_{1,2}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} + \frac{1}{2}(0.2)A_{1,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,2}\right] + \frac{1}{2}(0.2)B_{1,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{1,2}\left[A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right] - \left[A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} - A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,2}\right]$$

$$+\left[A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - A_{0,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right] - \left[A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} - A_{0,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,2}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{1,2}\left[B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right] - \left[B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} - B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$+\left[B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - B_{1,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right] - \left[B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} - B_{1,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{1,2}B_{1,2} + B_{1,2}A_{1,2}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,1}\right]$$

$$W_{1,2}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\left[\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]$$

$$+\left[\left[\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]\right]$$

$$+\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.0625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{1,2}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-0.0070\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0006\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0005\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{1,2}^{0}=\begin{pmatrix}0.0458\\-0.0070\\0.0000\end{pmatrix}$$

then,

$$\begin{pmatrix}Z_{1,2}\\U_{1,2}\\V_{1,2}\end{pmatrix}=\begin{pmatrix}0.0458\\-0.0070\\0.0000\end{pmatrix}$$

$$Let \quad n=0, \quad l=1, \quad m=3$$

Then we obtain $\boldsymbol{W}_{l,m}^{n} = \begin{pmatrix} Z_{1,3} \\ U_{1,3} \\ V_{1,3} \end{pmatrix}$, $W_{l,m}^{n} = W_{1,3}^{0} = \begin{pmatrix} Z_{1,3} \\ U_{1,3} \\ V_{1,3} \end{pmatrix} = \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix}$

$$W_{1,3}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} + \frac{1}{2}(0.2)A_{1,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,3}\right] + \frac{1}{2}(0.2)B_{1,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{1,3}\left[A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right] - \left[A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,3}\right]$$

$$+\left[A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - A_{0,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right] - \left[A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - A_{0,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,3}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{1,3}\left[B_{1,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,4} - B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right] - \left[B_{1,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\left[B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,4} - B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right] - \left[B_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - B_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{1,3}B_{1,3} + B_{1,3}A_{1,3}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{0,2}\right]$$

$$W_{1,3}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]$$

$$+\left[\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.00625\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{1,3}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-0.0056\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0.0056\end{pmatrix}+\begin{pmatrix}-0.0005\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0006\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-0.00006\\0\end{pmatrix}$$

$$W_{1,3}^{0}=\begin{pmatrix}0.0341\\-0.00566\\0.00566\end{pmatrix}$$

then, $$\begin{pmatrix}Z_{1,3}\\U_{1,3}\\V_{1,3}\end{pmatrix}=\begin{pmatrix}0.0341\\-0.00566\\0.00566\end{pmatrix}$$

$$Let \quad n=0, \quad l=2, \quad m=1$$

Then we obtain $\quad \boldsymbol{W}_{l,m}^{n} = \begin{pmatrix} Z_{2,1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix}$, $\quad W_{l,m}^{n} = W_{2,1}^{0} = \begin{pmatrix} Z_{2,1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix}$

$$W_{2,1}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} + \frac{1}{2}(0.2)A_{2,1}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right] + \frac{1}{2}(0.2)B_{2,1}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,0}\right]$$

$$+\frac{1}{4}(0.2)^{2}A_{2,1}\left[A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right] - \left[A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} - A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$+\left[A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - A_{1,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right] - \left[A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} - A_{1,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$+\frac{1}{4}(0.2)^{2}B_{2,1}\left[B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right] - \left[B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} - B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,0}\right]$$

$$+\left[B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - B_{2,0}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right] - \left[B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1} - B_{2,0}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,0}\right]$$

$$+\frac{1}{8}(0.2)^{2}\left[A_{2,1}B_{2,1} + B_{2,1}A_{2,1}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,0} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,0}\right]$$

$$W_{2,1}^{0} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} + 0.1 \begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix} \right] + 0.1 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \left[\begin{pmatrix} 0.0625 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$+0.01 \begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\left[\begin{bmatrix} 0 & -1 & 0 \\ -1.1875 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} \right] - \left[\begin{bmatrix} 0 & -1 & 0 \\ -1.1875 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} \right] \right.$$

$$\left. + \left[\begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ -1.0625 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} \right] - \left[\begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ -1.0625 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix} \right] \right]$$

$$+0.01 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \left[\left[\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.25 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0625 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} \right] - \left[\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.25 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \right.$$

$$\left. + \left[\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0625 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} \right] - \left[\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \right]$$

$$+0.005 \left[\begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.125 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1.125 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] \left[\begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right]$$

We got,

$$W_{2,1}^{0} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -0.0070 \end{pmatrix} + \begin{pmatrix} -0.0004 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -0.0006 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$W_{2,1}^{0} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

then,

$$\begin{pmatrix} Z_{2,1} \\ U_{2,1} \\ V_{2,1} \end{pmatrix} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$Let \quad n=0, \quad l=2, \quad m=2$$

Then we obtain $W_{l,m}^{n} = \begin{pmatrix} Z_{2,2} \\ U_{2,2} \\ V_{2,2} \end{pmatrix}$, $W_{l,m}^{n} = W_{2,2}^{0} = \begin{pmatrix} Z_{2,2} \\ U_{2,2} \\ V_{2,2} \end{pmatrix} = \begin{pmatrix} 0.0625 \\ 0 \\ 0 \end{pmatrix}$

$$W_{2,2}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} + \frac{1}{2}(0.2)A_{2,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right] + \frac{1}{2}(0.2)B_{2,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{2,2}\left[A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right] - \left[A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\left[A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right] - \left[A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - A_{1,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{2,2}\left[B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right] - \left[B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$+\left[B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right] - \left[B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - B_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{2,2}B_{2,2} + B_{2,2}A_{2,2}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,1}\right]$$

$$W_{2,2}^{0}=\begin{pmatrix}0.0625\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.125&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{2,2}^{0}=\begin{pmatrix}0.0625\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0004\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0008\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{2,2}^{0}=\begin{pmatrix}0.0613\\0.0000\\0.0000\end{pmatrix}$$

then,

$$\begin{pmatrix}Z_{2,2}\\U_{2,2}\\V_{2,2}\end{pmatrix}=\begin{pmatrix}0.0613\\0.0000\\0.0000\end{pmatrix}$$

$$Let \quad n=0, \quad l=2, \quad m=3$$

Then we obtain $W_{l,m}^{n} = \begin{pmatrix} Z_{2,3} \\ U_{2,3} \\ V_{2,3} \end{pmatrix}$, $W_{l,m}^{n} = W_{2,3}^{0} = \begin{pmatrix} Z_{2,3} \\ U_{2,3} \\ V_{2,3} \end{pmatrix} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix}$

$$W_{2,3}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} + \frac{1}{2}(0.2)A_{2,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right] + \frac{1}{2}(0.2)B_{2,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{2,3}\left[A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right] - \left[A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$+\left[A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right] - \left[A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - A_{1,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,3}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{2,3}\left[B_{2,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right] - \left[B_{2,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$+\left[B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right] - \left[B_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - B_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{2,3}B_{2,3} + B_{2,3}A_{2,3}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{1,2}\right]$$

$$W_{2,3}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.25&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{2,3}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0.0086\end{pmatrix}+\begin{pmatrix}-0.0002\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0009\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{2,3}^{0}=\begin{pmatrix}0.0458\\0.0000\\0.0086\end{pmatrix}$$

then,

$$\begin{pmatrix}Z_{2,3}\\U_{2,3}\\V_{2,3}\end{pmatrix}=\begin{pmatrix}0.0458\\0.0000\\0.0000\end{pmatrix}$$

$$Let \quad n=0, \quad l=3, \quad m=1$$

Then we obtain $\quad W_{l,m}^{n} = \begin{pmatrix} Z_{3,1} \\ U_{3,1} \\ V_{3,1} \end{pmatrix}$, $\quad W_{l,m}^{n} = W_{3,1}^{0} = \begin{pmatrix} Z_{3,1} \\ U_{3,1} \\ V_{3,1} \end{pmatrix} = \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix}$

$$W_{3,1}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} + \frac{1}{2}(0.2)A_{3,1}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,1} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right] + \frac{1}{2}(0.2)B_{3,1}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,0}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{3,1}\left[A_{4,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,1} - A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right] - \left[A_{4,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$+\left[A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,1} - A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right] - \left[A_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - A_{2,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{3,1}\left[B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right] - \left[B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right]$$

$$+\left[B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - B_{3,0}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right] - \left[B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1} - B_{3,0}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,0}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{3,1}B_{3,1} + B_{3,1}A_{3,1}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,0} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,0}\right]$$

$$W_{3,1}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\left[\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.125&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.1875&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0625\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{3,1}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0.0056\\0\end{pmatrix}+\begin{pmatrix}0\\0\\-0.0056\end{pmatrix}+\begin{pmatrix}-0.0007\\0\\0\end{pmatrix}+\begin{pmatrix}0.0008\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{3,1}^{0}=\begin{pmatrix}0.03490\\0.0056\\-0.0056\end{pmatrix}$$

then,

$$\begin{pmatrix}Z_{3,1}\\U_{3,1}\\V_{3,1}\end{pmatrix}=\begin{pmatrix}0.03490\\0.0056\\-0.0056\end{pmatrix}$$

$$Let \quad n=0, \quad l=3, \quad m=2$$

Then we obtain $W_{l,m}^{n} = \begin{pmatrix} Z_{3,2} \\ U_{3,2} \\ V_{3,2} \end{pmatrix}$, $W_{l,m}^{n} = W_{3,2}^{0} = \begin{pmatrix} Z_{3,2} \\ U_{3,2} \\ V_{3,2} \end{pmatrix} = \begin{pmatrix} 0.0469 \\ 0 \\ 0 \end{pmatrix}$

$$W_{3,2}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} + \frac{1}{2}(0.2)A_{3,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right] + \frac{1}{2}(0.2)B_{3,2}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{3,2}\left[A_{4,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,2} - A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right] - \left[A_{4,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$+\left[A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,2} - A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right] - \left[A_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - A_{2,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{3,2}\left[B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right] - \left[B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right]$$

$$+\left[B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right] - \left[B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2} - B_{3,1}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,1}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{3,2}B_{3,2} + B_{3,2}A_{3,2}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,1} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,1}\right]$$

$$W_{3,2}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+0.1\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]+0.1\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\left[\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\left[\begin{bmatrix}0&-1&0\\-1.5&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.5&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]$$

$$+\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.25&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]$$

$$+\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.1875&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]$$

$$+0.005\left[\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]$$

We got,

$$W_{3,2}^{0}=\begin{pmatrix}0.0469\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0.0086\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0009\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0006\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{3,2}^{0}=\begin{pmatrix}0.0454\\0.0086\\0.0000\end{pmatrix}$$

then,

$$\begin{pmatrix}Z_{3,2}\\U_{3,2}\\V_{3,2}\end{pmatrix}=\begin{pmatrix}0.0454\\0.0086\\0.0000\end{pmatrix}$$

$$Let \quad n=0, \quad l=3, \quad m=3$$

Then we obtain $W_{l,m}^{n} = \begin{pmatrix} Z_{3,3} \\ U_{3,3} \\ V_{3,3} \end{pmatrix}$, $W_{l,m}^{n} = W_{3,3}^{0} = \begin{pmatrix} Z_{3,3} \\ U_{3,3} \\ V_{3,3} \end{pmatrix} = \begin{pmatrix} 0.0352 \\ 0 \\ 0 \end{pmatrix}$

$$W_{3,3}^{0} = \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} + \frac{1}{2}(0.2)A_{3,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,3} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right] + \frac{1}{2}(0.2)B_{3,3}\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right]$$

$$+\frac{1}{4}(0.2)^2 A_{3,3}\left[A_{4,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,3} - A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3}\right] - \left[A_{4,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right]$$

$$+\left[A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,3} - A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3}\right] - \left[A_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - A_{2,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,3}\right]$$

$$+\frac{1}{4}(0.2)^2 B_{3,3}\left[B_{3,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,4} - B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3}\right] - \left[B_{3,4}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right]$$

$$+\left[B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,4} - B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3}\right] - \left[B_{3,3}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,3} - B_{3,2}\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{3,2}\right]$$

$$+\frac{1}{8}(0.2)^2\left[A_{3,3}B_{3,3} + B_{3,3}A_{3,3}\right]\left[\begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{4,2} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,4} - \begin{pmatrix} Z \\ U \\ V \end{pmatrix}_{2,2}\right]$$

$$W_{3,3}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+0.1\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]+0.1\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&-1&0\\-1.75&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.75&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&-1&0\\-1.375&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.01\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\left[\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.75&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.75&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right.$$

$$\left.+\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}\right]-\left[\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{pmatrix}0.0352\\0\\0\end{pmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\-1.375&0&0\end{bmatrix}\begin{pmatrix}0.0469\\0\\0\end{pmatrix}\right]\right]$$

$$+0.005\left[\left[\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&0\\-1.5625&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1.5625&0&0\\0&0&0\end{bmatrix}\right]\left[\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}-\begin{pmatrix}0.0625\\0\\0\end{pmatrix}\right]\right]$$

We got,

$$W_{3,3}^{0}=\begin{pmatrix}0.0352\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0.0073\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0.0073\end{pmatrix}+\begin{pmatrix}-0.0009\\0\\0\end{pmatrix}+\begin{pmatrix}-0.0008\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$W_{3,3}^{0}=\begin{pmatrix}0.0335\\0.0073\\0.0073\end{pmatrix}$$

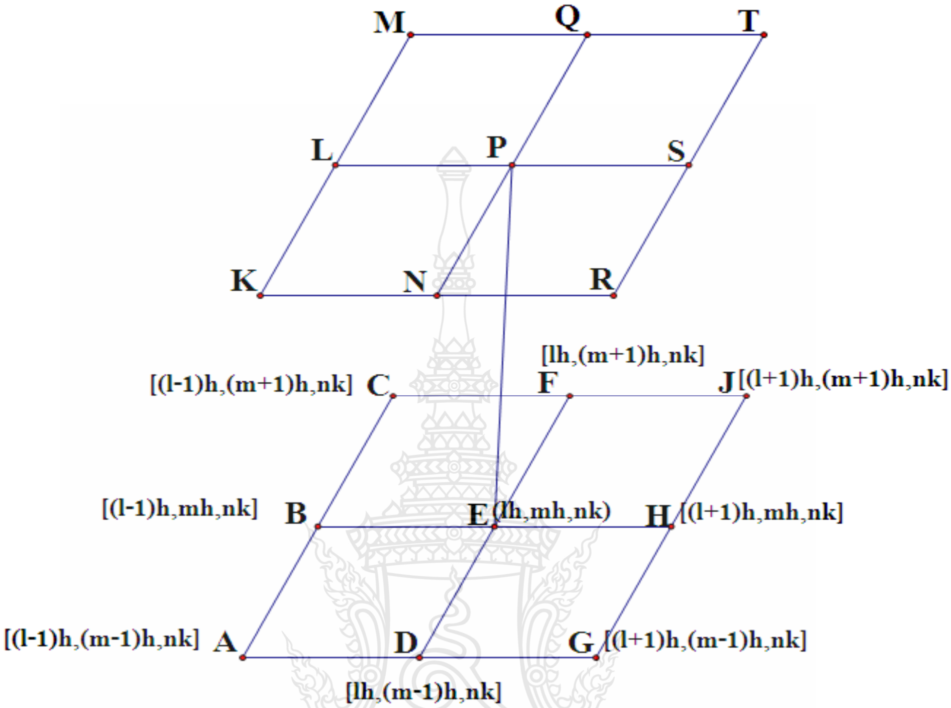
then,

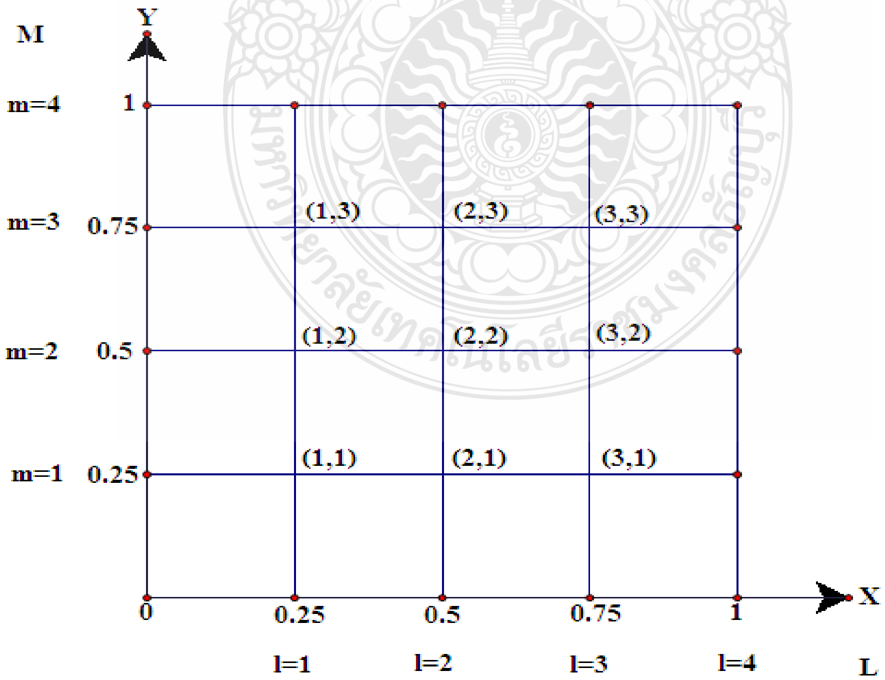
$$\begin{pmatrix}Z_{3,3}\\U_{3,3}\\V_{3,3}\end{pmatrix}=\begin{pmatrix}0.0335\\0.0073\\0.0073\end{pmatrix}$$

Lax (3, 0.1)

$$\Delta h = \Delta k = 0.25, \quad \Delta t = 0.05$$

$$p = \frac{\Delta t}{\Delta y} = \frac{0.05}{0.25} = 0.2$$





$$X_1 = l\Delta x$$
$$X_1 = 1(0.25) = 0.25$$
$$X_2 = 2(0.25) = 0.50$$
$$X_3 = 3(0.25) = 0.75$$
$$X_4 = 4(0.25) = 1.00$$

Lax (3, 0.1)

$\Delta t = 0.05$

	$W^1_{1,1}$	$W^1_{1,2}$	$W^1_{1,3}$	$W^1_{2,1}$	$W^1_{2,2}$	$W^1_{2,3}$	$W^1_{3,1}$	$W^1_{3,2}$	$W^1_{3,3}$
Z	0.03324	0.04360	0.03315	0.04371	0.05686	0.04335	0.03387	0.04301	0.03259
ζ	0.03324	0.04360	0.03315	0.04371	0.05686	0.04335	0.03387	0.04301	0.03259
U	-0.00966	-0.01313	-0.01029	-0.00006	0.00009	0.00010	0.01024	0.01466	0.01186
u	-0.03024	-0.04110	-0.03221	-0.00019	0.00028	0.00031	0.03206	0.04589	0.03713
V	-0.00965	0.00002	0.01029	-0.01313	0.00006	0.01466	-0.01019	0.00018	0.01181
v	-0.03021	0.00006	0.03221	-0.04110	0.00019	0.04589	-0.03190	0.00056	0.03697

$Z = \dfrac{\zeta}{h}$ $\quad \zeta = Zh$ $\quad U = \dfrac{u}{\sqrt{gh}}$ $\quad u = U\sqrt{gh}$ $\quad u = U\sqrt{(9.8)(1)}$

$V = \dfrac{v}{\sqrt{gh}}$ $\quad v = V\sqrt{gh}$ $\quad v = V\sqrt{(9.8)(1)}$

	$W^1_{1,3}$	$W^1_{2,3}$	$W^1_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

ζ

	0.03315	0.04335	0.03259	
	0.04360	0.05686	0.04301	
	0.03324	0.04371	0.03387	

	$W_{1,3}$	$W_{2,3}$	$W_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

u

0	0	0	0	0
-0.03221	-0.03221	0.00031	0.03713	0.03713
-0.04110	-0.04110	0.00028	0.04589	0.04589
-0.03024	-0.03024	-0.00019	0.03206	0.03206
0	0	0	0	0

	$W_{1,3}$	$W_{2,3}$	$W_{3,3}$	
	$W^1_{1,2}$	$W^1_{2,2}$	$W^1_{3,2}$	
	$W^1_{1,1}$	$W^1_{2,1}$	$W^1_{3,1}$	

v

0	0.03221	0.04589	0.03697	0
0	0.03221	0.04589	0.03697	0
0	0.00006	0.00019	0.00056	0
0	-0.03021	-0.04110	-0.03190	0
0	-0.03021	-0.04110	-0.03190	0

$$l = 1, \quad m = 1, \quad X = l\Delta x, \quad Y = m\Delta y$$

$$A_{1,1} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+XY) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+lm(\Delta x)^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -(1+(1)(1)(0.25)^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{1,1} = \begin{bmatrix} 0 & -1 & 0 \\ -1.0625 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

	1	Δ_x	X=LΔX	m	Δy	Y=mΔY	-(1+XY)
$A_{1,1}$	1	0.25	0.25	1	0.25	0.25	-1.0625
$A_{0,1}$	0	0.25	0	1	0.25	0.25	-1.0000
$A_{0,2}$	0	0.25	0	2	0.25	0.5	-1.0000
$A_{0,3}$	0	0.25	0	3	0.25	0.75	-1.0000
$A_{0,4}$	0	0.25	0	4	0.25	1	-1.0000
$A_{1,2}$	1	0.25	0.25	2	0.25	0.5	-1.1250
$A_{1,3}$	1	0.25	0.25	3	0.25	0.75	-1.1875
$A_{1,4}$	1	0.25	0.25	4	0.25	1	-1.2500
$A_{2,1}$	2	0.25	0.5	1	0.25	0.25	-1.1250
$A_{2,2}$	2	0.25	0.5	2	0.25	0.5	-1.2500
$A_{2,3}$	2	0.25	0.5	3	0.25	0.75	-1.3750
$A_{2,4}$	2	0.25	0.5	4	0.25	1	-1.5000
$A_{3,1}$	3	0.25	0.75	1	0.25	0.25	-1.1875
$A_{3,2}$	3	0.25	0.75	2	0.25	0.5	-1.3750
$A_{3,3}$	3	0.25	0.75	3	0.25	0.75	-1.5625
$A_{3,4}$	3	0.25	0.75	4	0.25	1	-1.7500
$A_{4,1}$	4	0.25	1	1	0.25	0.25	-1.2500
$A_{4,2}$	4	0.25	1	2	0.25	0.5	-1.5000
$A_{4,3}$	4	0.25	1	3	0.25	0.75	-1.7500
$A_{4,4}$	4	0.25	1	4	0.25	1	-2.0000

$$l = 1, \quad m = 1, \quad X = l\Delta x, \quad Y = m\Delta y$$

$$B_{1,1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+XY) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+lm(\Delta x)^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -(1+(1)(1)(0.25) & 0 & 0 \end{bmatrix}$$

$$B_{1,1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1.0625 & 0 & 0 \end{bmatrix}$$

	l	Δ_x	X=LΔX	m	Δy	Y=mΔY	-(1+XY)
$B_{1,1}$	1	0.25	0.25	1	0.25	0.25	-1.0625
$B_{0,1}$	0	0.25	0	1	0.25	0.25	-1.0000
$B_{0,2}$	0	0.25	0	2	0.25	0.5	-1.0000
$B_{0,3}$	0	0.25	0	3	0.25	0.75	-1.0000
$B_{0,4}$	0	0.25	0	4	0.25	1	-1.0000
$B_{1,2}$	1	0.25	0.25	2	0.25	0.5	-1.1250
$B_{1,3}$	1	0.25	0.25	3	0.25	0.75	-1.1875
$B_{1,4}$	1	0.25	0.25	4	0.25	1	-1.2500
$B_{2,1}$	2	0.25	0.5	1	0.25	0.25	-1.1250
$B_{2,2}$	2	0.25	0.5	2	0.25	0.5	-1.2500
$B_{2,3}$	2	0.25	0.5	3	0.25	0.75	-1.3750
$B_{2,4}$	2	0.25	0.5	4	0.25	1	-1.5000
$B_{3,1}$	3	0.25	0.75	1	0.25	0.25	-1.1875
$B_{3,2}$	3	0.25	0.75	2	0.25	0.5	-1.3750
$B_{3,3}$	3	0.25	0.75	3	0.25	0.75	-1.5625
$B_{3,4}$	3	0.25	0.75	4	0.25	1	-1.7500
$B_{4,1}$	4	0.25	1	1	0.25	0.25	-1.2500
$B_{4,2}$	4	0.25	1	2	0.25	0.5	-1.5000
$B_{4,3}$	4	0.25	1	3	0.25	0.75	-1.7500
$B_{4,4}$	4	0.25	1	4	0.25	1	-2.0000

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+\frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)]$$

$$l = 1, m = 1, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(1-1)0.25, (1-1)0.25, 1(0.25)] = (0,0,0.25) = W_{0,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(1-1)0.25, (1)0.25, 1(0.25)] = (0, 0.25, 0.25) = W_{0,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(1-1)0.25, (1+1)0.25, 1(0.25)] = (0, 0.5, 0.25) = W_{0,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(1)0.25, (1-1)0.25, 1(0.25)] = (0.25, 0, 0.25) = W_{1,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E[lh, mh, nk] = [(1)0.25, (1)0.25, 1(0.25)] = (0.25, 0.25, 0.25) = W_{1,1} = \begin{pmatrix} 0.0342 \\ -0.0050 \\ -0.0050 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(1)0.25, (1+1)0.25, 1(0.25)] = (0.25, 0.5, 0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(1+1)0.25, (1-1)0.25, 1(0.25)] = (0.5, 0, 0.25) = W_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(1+1)0.25, (1)0.25, 1(0.25)] = (0.5, 0, 25, 0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0 \\ -0.0070 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(1+1)0.25, (1+1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{1,1} = [I - p^2(A^2 + B^2)U_{1,1} + \frac{1}{2}pA(I + pA)U_{2,1} - \frac{1}{2}pA(I - pA)U_{0,1}$$

$$+\frac{1}{2}pB(I + pB)U_{1,2} - \frac{1}{2}pB(I - pB)U_{1,0} + \frac{1}{8}p^2(AB + BA)(U_{2,2} - U_{2,0} - U_{0,2} + U_{0,0})]$$

Step 3 t=0.1

$\Delta t = 0.05$

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0			

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

$U_{1,1}$ = I − 0.04 (A^2 + B^2)

I				$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0		2.125	0	0	0.085	0	0	0.915	0	0	0.0342	0.03129
0	1	0	- 0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	-0.005	-0.00479
0	0	1		0	0	1.0625	0	0	0.0425	0	0	0.9575	-0.005	-0.00479

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0459	0.000975
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	-0.004877
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	-0.007	0

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0	0
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0458	0.000973
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	-0.007	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	-0.004866

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	0

$\frac{1}{8}p^2$ = 0.005

AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0	0	0	0	0	0	0	0	0	0.0613	0	0	0	0.0613	0
0	0	1.0625	0	0	0	0	0	0.005313	0	0	0	0	0	0
0	0	0	0	1.0625	0	0	0.005313	0	0	0	0	0	0	0

$U_{1,1} =$

0.03324
-0.00966
-0.00965

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+\frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)\,]$$

$$l = 1, m = 2, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(1-1)0.25,\ (2-1)0.25, 1(0.25)] = (0, 0.25, 0.25) = W_{0,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(1-1)0.25,\ (2)0.25, 1(0.25)] = (0, 0.5, 0.25) = W_{0,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(1-1)0.25,\ (2+1)0.25, 1(0.25)] = (0, 0.75, 0.25) = W_{0,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(1)0.25,\ (2-1)0.25, 1(0.25)] = (0.25, 0.25, 0.25) = W_{1,1} = \begin{pmatrix} 0.0342 \\ -0.0050 \\ -0.0050 \end{pmatrix}$$

$$E[lh, mh, nk] = [(1)0.25,\ (2)0.25, 1(0.25)] = (0.25, 0.5, 0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(1)0.25,\ (2+1)0.25, 1(0.25)] = (0.25, 0.75, 0.25) = W_{1,3} = \begin{pmatrix} 0.0341 \\ -0.00566 \\ 0.00566 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(1+1)0.25,\ (2-1)0.25, 1(0.25)] = (0.5, 0.25, 0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(1+1)0.25,\ (2)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(1+1)0.25,\ (2+1)0.25, 1(0.25)] = (0.5, 0.75, 0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$U_{1,2} = [I - p^2(A^2 + B^2)U_{1,2} + \frac{1}{2}pA(I + pA)U_{2,2} - \frac{1}{2}pA(I - pA)U_{0,2}$$

$$+\frac{1}{2}pB(I + pB)U_{1,3} - \frac{1}{2}pB(I - pB)U_{1,1} + \frac{1}{8}p^2(AB + BA)(U_{2,3} - U_{2,1} - U_{0,3} + U_{0,1})\,]$$

Step 3 t=0.1

$\Delta t = 0.05$

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0	0	0	0

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

$U_{1,2}$

I				P^2	A^2+B^2			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0			2.125	0	0	0.085	0	0	0.915	0	0	0.0458	0.04191
0	1	0	-	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	-0.007	-0.00670
0	0	1			0	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.00000

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0613	0.001303
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	-0.006513
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0	0
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0341	0.000159
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	-0.00566	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0.00566	-0.003503

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0342	-0.000227
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	-0.005	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	-0.005	-0.003528

$\frac{1}{8}p^2$	AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
	0	0	0	0	0	0	0	0	0	0.0458	0.0459	0	0	-0.0001	0
0.005	0	0	1.0625	0	0	0	0	0	0.005313	0	0	0	0	0	8.29E-05
	0	0	0	0	1.0625	0	0	0.005313	0	0.0086	-0.007	0	0	0.0156	0

$U_{1,2} =$

0.04360
-0.01313
0.00002

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2} pA(I + pA)U_H - \frac{1}{2} pA(I - pA)U_B$$

$$+ \frac{1}{2} pB(I + pB)U_F - \frac{1}{2} pB(I - pB)U_D + \frac{1}{8} p^2(AB + BA)(U_J - U_G - U_C + U_A)\ ]$$

$$l = 1, m = 3, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h, (m-1)h, nk] = [(1-1)0.25, (3-1)0.25, 1(0.25)] = (0, 0.5, 0.25) = W_{0,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(1-1)0.25, (3)0.25, 1(0.25)] = (0, 0.75, 0.25) = W_{0,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C[(l-1)h, (m+1)h, nk] = [(1-1)0.25, (3+1)0.25, 1(0.25)] = (0, 0.75, 0.25) = W_{0,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D[lh, (m-1)h, nk] = [(1)0.25, (3-1)0.25, 1(0.25)] = (0.25, 0.5, 0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$E[lh, mh, nk] = [(1)0.25, (3)0.25, 1(0.25)] = (0.25, 0.75, 0.25) = W_{1,3} = \begin{pmatrix} 0.0341 \\ -0.00566 \\ 0.00566 \end{pmatrix}$$

$$F[lh, (m+1)h, nk] = [(1)0.25, (3+1)0.25, 1(0.25)] = (0.25, 1.0, 0.25) = W_{1,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$G[(l+1)h, (m-1)h, nk] = [(1+1)0.25, (3-1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(1+1)0.25, (3)0.25, 1(0.25)] = (0.5, 0.75, 0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$J[(l+1)h, (m+1)h, nk] = [(1+1)0.25, (3+1)0.25, 1(0.25)] = (0.5, 1.0, 0.25) = W_{2,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{1,3} = [I - p^2(A^2 + B^2)U_{1,3} + \frac{1}{2} pA(I + pA)U_{2,3} - \frac{1}{2} pA(I - pA)U_{0,3}$$

$$+ \frac{1}{2} pB(I + pB)U_{1,4} - \frac{1}{2} pB(I - pB)U_{1,2} + \frac{1}{8} p^2(AB + BA)(U_{2,4} - U_{2,2} - U_{0,4} + U_{0,2})\ ]$$

Step 3 t=0.1

$\Delta t = 0.05$

A × A = A^2

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0			

A × B = AB

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B × B = B^2

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

B × A = BA

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

$U_{1,3}$

I			-	A^2 + B^2			
1	0	0			2.125	0	0
0	1	0	-	0.04	0	1.0625	0
0	0	1			0	0	1.0625

$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
0.085	0	0	0.915	0	0	0.0341	0.03120
0	0.0425	0	0	0.9575	0	-0.00566	-0.00542
0	0	0.0425	0	0	0.9575	0.00566	0.00542

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0458	0.000973
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	-0.004866
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0.0086	0

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0	0
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0458	-0.000973
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	-0.007	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	-0.004866

$\frac{1}{8}p^2$ = 0.005

AB			BA			$\frac{1}{8}p^2(AB+BA)$		
0	0	0	0	0	0	0	0	0
0	0	1.0625	0	0	0	0	0	0.005313
0	0	0	0	1.0625	0	0	0.005313	0

U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0	0.0613	0	0	-0.0613	0
0	0	0	0	0	0
0	0	0	0	0	0

$U_{1,3} =$

0.03315
-0.01029
0.01029

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2} pA(I + pA)U_H - \frac{1}{2} pA(I - pA)U_B$$

$$+\frac{1}{2} pB(I + pB)U_F - \frac{1}{2} pB(I - pB)U_D + \frac{1}{8} p^2(AB + BA)(U_J - U_G - U_C + U_A)\ ]$$

$$l = 2,\ m = 1,\ n = 1,\ \Delta x = 0.25,\ \Delta y = 0.25$$

$$A[(l-1)h, (m-1)h, nk] = [(2-1)0.25,\ (1-1)0.25, 1(0.25)] = (0.25, 0, 0.25) = W_{1,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(2-1)0.25,\ (1)0.25, 1(0.25)] = (0.25, 0.25, 0.25) = W_{1,1} = \begin{pmatrix} 0.0342 \\ -0.0050 \\ -0.0050 \end{pmatrix}$$

$$C[(l-1)h, (m+1)h, nk] = [(2-1)0.25,\ (1+1)0.25, 1(0.25)] = (0.25, 0.5, 0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$D[lh, (m-1)h, nk] = [(2)0.25,\ (1-1)0.25, 1(0.25)] = (0.5, 0, 0.25) = W_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E[lh, mh, nk] = [(2)0.25,\ (1)0.25, 1(0.25)] = (0.5, 0.25, 0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$F[lh, (m+1)h, nk] = [(2)0.25,\ (1+1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$G[(l+1)h, (m-1)h, nk] = [(2+1)0.25,\ (1-1)0.25, 1(0.25)] = (0.75, 0, 0.25) = W_{3,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(2+1)0.25,\ (1)0.25, 1(0.25)] = (0.75, 0.25, 0.25) = W_{3,1} = \begin{pmatrix} 0.03490 \\ 0.0056 \\ -0.0056 \end{pmatrix}$$

$$J[(l+1)h, (m+1)h, nk] = [(2+1)0.25,\ (1+1)0.25, 1(0.25)] = (0.75, 0.5, 0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$U_{2,1} = [I - p^2(A^2 + B^2)U_{2,1} + \frac{1}{2} pA(I + pA)U_{3,1} - \frac{1}{2} pA(I - pA)U_{1,1}$$

$$+\frac{1}{2} pB(I + pB)U_{2,2} - \frac{1}{2} pB(I - pB)U_{2,0} + \frac{1}{8} p^2(AB + BA)(U_{3,2} - U_{3,0} - U_{1,2} + U_{1,0})\ ]$$

Step 3 t=0.1

$\Delta t = 0.05$

A × A = A^2

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0	0	0	0

A × B = AB

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B × B = B^2

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

B × A = BA

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

$U_{2,1}$ = I − 0.04 ($A^2 + B^2$)

I				p²	$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0			2.125	0	0	0.085	0	0	0.915	0	0	0.0459	0.04200
0	1	0	-	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	0	0.00000
0	0	1			0	0	1.0625	0	0	0.0425	0	0	0.9575	-0.007	-0.00670

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0349	0.000182
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0.0056	-0.003589
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	-0.0056	0

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0342	-0.000227
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	-0.005	-0.003528
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	-0.005	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0613	0.001303
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	-0.006513

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	0

$\frac{1}{8}p^2$ = 0.005

AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0	0	0	0	0	0	0	0	0	0.0454	0	0.0458	0	-0.0004	0
0	0	1.0625	0	0	0	0	0	0.005313	0.0086	0	-0.007	0	0.0156	0
0	0	0	0	1.0625	0	0	0.005313	0	0	0	0	0	0	8.29E-05

$U_{2,1} =$

0.04371
-0.00006
-0.01313

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+ \frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)]$$

$$l = 2, m = 2, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(2-1)0.25,\ (2-1)0.25,1(0.25)] = (0.25,0.25,0.25) = W_{1,1} = \begin{pmatrix} 0.0342 \\ -0.0050 \\ -0.0050 \end{pmatrix}$$

$$B[(l-1)h,mh,nk] = [(2-1)0.25,\ (2)0.25,1(0.25)] = (0.25,0.5,0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(2-1)0.25,\ (2+1)0.25,1(0.25)] = (0.25,0.75,0.25) = W_{1,3} = \begin{pmatrix} 0.0341 \\ -0.00566 \\ 0.00566 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(2)0.25,\ (2-1)0.25,1(0.25)] = (0.5,0.25,0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$E[lh,mh,nk] = [(2)0.25,\ (2)0.25,1(0.25)] = (0.5,0.5,0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(2)0.25,\ (2+1)0.25,1(0.25)] = (0.5,0.75,0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(2+1)0.25,\ (2-1)0.25,1(0.25)] = (0.75,0.25,0.25) = W_{3,1} = \begin{pmatrix} 0.0349 \\ 0.0056 \\ -0.0056 \end{pmatrix}$$

$$H[(l+1)h,mh,nk] = [(2+1)0.25,\ (2)0.25,1(0.25)] = (0.75,0.5,0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(2+1)0.25,\ (2+1)0.25,1(0.25)] = (0.75,0.75,0.25) = W_{3,3} = \begin{pmatrix} 0.0335 \\ 0.0073 \\ 0.0073 \end{pmatrix}$$

$$U_{2,2} = [I - p^2(A^2 + B^2)U_{2,2} + \frac{1}{2}pA(I + pA)U_{3,2} - \frac{1}{2}pA(I - pA)U_{1,2}$$

$$+ \frac{1}{2}pB(I + pB)U_{2,3} - \frac{1}{2}pB(I - pB)U_{2,1} + \frac{1}{8}p^2(AB + BA)(U_{3,3} - U_{3,1} - U_{1,3} + U_{1,1})]$$

Step 3 t=0.1

$\Delta t = 0.05$

A · A = A^2

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0			

A · B = AB

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B · A = BA

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

B · B = B^2

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

$U_{2,2}$ = I − 0.04 (A^2 + B^2) U_E

I			$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0	2.125	0	0	0.085	0	0	0.915	0	0	0.0613	0.05609
0	1	0	0	1.0625	0	0	0.0425	0	0	0.9575	0	0	0.00000
0	0	1	0	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.00000

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0454	0.000105
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0.0086	-0.004641
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0458	-0.000273
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	-0.007	-0.004718
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0458	0.000113
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0.0086	-0.004684

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0459	-0.000275
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	-0.007	-0.004728

$\frac{1}{8}p^2$ = 0.005

AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0	0	0	0	0	0	0	0	0	0.0335	0.0349	0.0341	0.0342	-0.0013	0
0	0	1.0625	0	0	0	0	0	0.005313	0.0073	0.0056	-0.00566	-0.005	0.00236	1.19E-05
0	0	0	0	1.0625	0	0	0.005313	0	0.0073	-0.0056	0.00566	-0.005	0.00224	1.25E-05

$U_{2,2} =$

0.05686
0.00009
0.00006

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2} pA(I + pA)U_H - \frac{1}{2} pA(I - pA)U_B$$

$$+\frac{1}{2} pB(I + pB)U_F - \frac{1}{2} pB(I - pB)U_D + \frac{1}{8} p^2(AB + BA)(U_J - U_G - U_C + U_A)\,]$$

$$l = 2, m = 3, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(2-1)0.25,\ (3-1)0.25, 1(0.25)] = (0.25, 0.5, 0.25) = W_{1,2} = \begin{pmatrix} 0.0458 \\ -0.0070 \\ 0.0000 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(2-1)0.25,\ (3)0.25, 1(0.25)] = (0.25, 0.75, 0.25) = W_{1,3} = \begin{pmatrix} 0.0341 \\ -0.00566 \\ 0.00566 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(2-1)0.25,\ (3+1)0.25, 1(0.25)] = (0.25, 1.0, 0.25) = W_{1,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(2)0.25,\ (3-1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$E[lh, mh, nk] = [(2)0.25,\ (3)0.25, 1(0.25)] = (0.5, 0.75, 0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(2)0.25,\ (3+1)0.25, 1(0.25)] = (0.5, 1.0, 0.25) = W_{2,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(2+1)0.25,\ (3-1)0.25, 1(0.25)] = (0.75, 0.5, 0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(2+1)0.25,\ (3)0.25, 1(0.25)] = (0.75, 0.75, 0.25) = W_{3,3} = \begin{pmatrix} 0.0335 \\ 0.0073 \\ 0.0073 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(2+1)0.25,\ (3+1)0.25, 1(0.25)] = (0.75, 1.0, 0.25) = W_{3,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{2,3} = [I - p^2(A^2 + B^2)U_{2,3} + \frac{1}{2} pA(I + pA)U_{3,3} - \frac{1}{2} pA(I - pA)U_{1,3}$$

$$+\frac{1}{2} pB(I + pB)U_{2,4} - \frac{1}{2} pB(I - pB)U_{2,2} + \frac{1}{8} p^2(AB + BA)(U_{3,4} - U_{3,2} - U_{1,4} + U_{1,2})\,]$$

Step 3 t=0.1

$\Delta t = 0.05$

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0	0	0	0

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

$U_{2,3}$

I			-		$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0			2.125	0	0	0.085	0	0	0.915	0	0	0.0458	0.04191
0	1	0	-	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	0	0.00000
0	0	1			0	0	1.0625	0	0	0.0425	0	0	0.9575	0.0086	0.00823

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0.0335	-1.81E-05
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0.0073	-0.003404
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0.0073	0

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0341	-0.000159
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	-0.00566	-0.003503
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0.00566	0

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0	0
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	0

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0613	-0.001303
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	-0.006513

$\frac{1}{8}p^2$	AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
	0	0	0	0	0	0	0	0	0	0	0.0454	0	0.0458	0.0004	0
0.005	0	0	1.0625	0	0	0	0	0	0.005313	0	0.0086	0	-0.007	-0.0156	0
	0	0	0	0	1.0625	0	0	0.005313	0	0	0	0	0	0	-8.29E-05

$U_{2,3} =$

0.04335
0.00010
0.01466

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+\frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)\,]$$

$$l = 3, m = 1, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(3-1)0.25,\ (1-1)0.25, 1(0.25)] = (0.5, 0, 0.25) = W_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(3-1)0.25,\ (1)0.25, 1(0.25)] = (0.5, 0.25, 0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(3-1)0.25,\ (1+1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(3)0.25,\ (1-1)0.25, 1(0.25)] = (0.75, 0, 0.25) = W_{3,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E[lh, mh, nk] = [(3)0.25,\ (1)0.25, 1(0.25)] = (0.75, 0.25, 0.25) = W_{3,1} = \begin{pmatrix} 0.0349 \\ 0.0056 \\ -0.0056 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(3)0.25,\ (1+1)0.25, 1(0.25)] = (0.75, 0.5, 0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(3+1)0.25,\ (1-1)0.25, 1(0.25)] = (1.0, 0, 0.25) = W_{4,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(3+1)0.25,\ (1)0.25, 1(0.25)] = (1.0, 0.25, 0.25) = W_{4,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(3+1)0.25,\ (1+1)0.25, 1(0.25)] = (1.0, 0.5, 0.25) = W_{4,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{3,1} = [I - p^2(A^2 + B^2)U_{3,1} + \frac{1}{2}pA(I + pA)U_{4,1} - \frac{1}{2}pA(I - pA)U_{2,1}$$

$$+\frac{1}{2}pB(I + pB)U_{3,2} - \frac{1}{2}pB(I - pB)U_{3,0} + \frac{1}{8}p^2(AB + BA)(U_{4,2} - U_{4,0} - U_{2,2} + U_{2,0})\,]$$

Step 3 t=0.1

$\Delta t = 0.05$

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0	0	0	0

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

B			B			B^2		
0	0	-1				1.0625	0	0
0	0	0				0	0	0
-1.0625	0	0				0	0	1.0625

$U_{3,1}$

I			-		$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0			2.125	0	0	0.085	0	0	0.915	0	0	0.0349	0.03193
0	1	0	-	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.0056	0.00536
0	0	1			0	0	1.0625	0	0	0.0425	0	0	0.9575	-0.0056	-0.00536

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0	0
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0459	-0.000975
0.1	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	-0.004877
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	-0.007	0

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0454	0.000965
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.0086	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	-0.004824

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0	0
0.1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	0

$\frac{1}{8}p^2$	AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
	0	0	0	0	0	0	0	0	0	0	0	0.0613	0	-0.0613	0
0.005	0	0	1.0625	0	0	0	0	0	0.005313	0	0	0	0	0	0
	0	0	0	0	1.0625	0	0	0.005313	0	0	0	0	0	0	0

$U_{3,1} =$

0.03387
0.01024
-0.01019

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+\frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)]$$

$$l = 3, m = 2, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(3-1)0.25,\ (2-1)0.25, 1(0.25)] = (0.5, 0.25, 0.25) = W_{2,1} = \begin{pmatrix} 0.0459 \\ 0.0000 \\ -0.0070 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(3-1)0.25,\ (2)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(3-1)0.25,\ (2+1)0.25, 1(0.25)] = (0.5, 0.75, 0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(3)0.25,\ (2-1)0.25, 1(0.25)] = (0.75, 0.25, 0.25) = W_{3,1} = \begin{pmatrix} 0.0349 \\ 0.0056 \\ -0.0056 \end{pmatrix}$$

$$E[lh, mh, nk] = [(3)0.25,\ (2)0.25, 1(0.25)] = (0.75, 0.5, 0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(3)0.25,\ (2+1)0.25, 1(0.25)] = (0.75, 0.75, 0.25) = W_{3,3} = \begin{pmatrix} 0.0335 \\ 0.0073 \\ 0.0073 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(3+1)0.25,\ (2-1)0.25, 1(0.25)] = (1.0, 0.25, 0.25) = W_{4,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(3+1)0.25,\ (2)0.25, 1(0.25)] = (1.0, 0.5, 0.25) = W_{4,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(3+1)0.25,\ (2+1)0.25, 1(0.25)] = (1.0, 0.75, 0.25) = W_{4,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{3,2} = [I - p^2(A^2 + B^2)U_{3,2} + \frac{1}{2}pA(I + pA)U_{4,2} - \frac{1}{2}pA(I - pA)U_{2,2}$$

$$+\frac{1}{2}pB(I + pB)U_{3,3} - \frac{1}{2}pB(I - pB)U_{3,1} + \frac{1}{8}p^2(AB + BA)(U_{4,3} - U_{4,1} - U_{2,3} + U_{2,1})]$$

Step 3 t=0.1

$\Delta t = 0.05$

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0			

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

$U_{3,2}$

I					$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0			2.125	0	0	0.085	0	0	0.915	0	0	0.0454	0.04154
0	1	0	-	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.0086	0.00823
0	0	1			0	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.00000

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0.1	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0	0
	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$	A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0.1	0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0613	-0.001303
	-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	-0.006513
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0.1	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0.0335	-1.81E-05
	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.0073	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0.0073	-0.003404

$\frac{1}{2}p$	B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0.1	0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0349	-0.000182
	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.0056	0
	-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	-0.0056	-0.003589

$\frac{1}{8}p^2$	AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0.005	0	0	0	0	0	0	0	0	0	0	0	0.0458	0.0459	0.0001	0
	0	0	1.0625	0	0	0	0	0	0.005313	0	0	0	0	0	-8.29E-05
	0	0	0	0	1.0625	0	0	0.005313	0	0	0	0.0086	-0.007	-0.0156	0

$U_{3,2} =$

0.04301
0.01466
0.00018

$$U_p = [I - p^2(A^2 + B^2)U_E + \frac{1}{2}pA(I + pA)U_H - \frac{1}{2}pA(I - pA)U_B$$

$$+\frac{1}{2}pB(I + pB)U_F - \frac{1}{2}pB(I - pB)U_D + \frac{1}{8}p^2(AB + BA)(U_J - U_G - U_C + U_A)\,]$$

$$l = 3, m = 3, n = 1, \Delta x = 0.25, \Delta y = 0.25$$

$$A[(l-1)h,(m-1)h,nk] = [(3-1)0.25,\ (3-1)0.25, 1(0.25)] = (0.5, 0.5, 0.25) = W_{2,2} = \begin{pmatrix} 0.0613 \\ 0.0000 \\ 0.0000 \end{pmatrix}$$

$$B[(l-1)h, mh, nk] = [(3-1)0.25,\ (3)0.25, 1(0.25)] = (0.5, 0.75, 0.25) = W_{2,3} = \begin{pmatrix} 0.0458 \\ 0.0000 \\ 0.0086 \end{pmatrix}$$

$$C[(l-1)h,(m+1)h,nk] = [(3-1)0.25,\ (3+1)0.25, 1(0.25)] = (0.5, 1.0, 0.25) = W_{2,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D[lh,(m-1)h,nk] = [(3)0.25,\ (3-1)0.25, 1(0.25)] = (0.75, 0.5, 0.25) = W_{3,2} = \begin{pmatrix} 0.0454 \\ 0.0086 \\ 0.0000 \end{pmatrix}$$

$$E[lh, mh, nk] = [(3)0.25,\ (3)0.25, 1(0.25)] = (0.75, 0.75, 0.25) = W_{3,3} = \begin{pmatrix} 0.0335 \\ 0.0073 \\ 0.0073 \end{pmatrix}$$

$$F[lh,(m+1)h,nk] = [(3)0.25,\ (3+1)0.25, 1(0.25)] = (0.75, 1.0, 0.25) = W_{3,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$G[(l+1)h,(m-1)h,nk] = [(3+1)0.25,\ (3-1)0.25, 1(0.25)] = (1.0, 0.5, 0.25) = W_{4,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H[(l+1)h, mh, nk] = [(3+1)0.25,\ (3)0.25, 1(0.25)] = (1.0, 0.75, 0.25) = W_{4,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$J[(l+1)h,(m+1)h,nk] = [(3+1)0.25,\ (3+1)0.25, 1(0.25)] = (1.0, 1.0, 0.25) = W_{4,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$U_{3,3} = [I - p^2(A^2 + B^2)U_{3,3} + \frac{1}{2}pA(I + pA)U_{4,3} - \frac{1}{2}pA(I - pA)U_{2,3}$$

$$+\frac{1}{2}pB(I + pB)U_{3,4} - \frac{1}{2}pB(I - pB)U_{3,2} + \frac{1}{8}p^2(AB + BA)(U_{4,4} - U_{4,2} - U_{2,4} + U_{2,2})\,]$$

Step 3 t=0.1

$\Delta t = 0.05$

A × A = A^2

A			A			A^2		
0	-1	0	0	-1	0	1.0625	0	0
-1.0625	0	0	-1.0625	0	0	0	1.0625	0
0	0	0	0	0	0			

A × B = AB

A			B			AB		
0	-1	0	0	0	-1	0	0	0
-1.0625	0	0	0	0	0	0	0	1.0625
0	0	0	-1.0625	0	0	0	0	0

B × B = B^2

B			B^2		
0	0	-1	1.0625	0	0
0	0	0	0	0	0
-1.0625	0	0	0	0	1.0625

B × A = BA

B			A			BA		
0	0	-1	0	-1	0	0	0	0
0	0	0	-1.0625	0	0	0	0	0
-1.0625	0	0	0	0	0	0	1.0625	0

$U_{3,3}$

I			-	$A^2 + B^2$			$P^2(A^2+B^2)$			$[I-P^2(A^2+B^2)]$			U_E	$[I-P^2(A^2+B^2)]U_E$
1	0	0		2.125	0	0	0.085	0	0	0.915	0	0	0.0335	0.03065
0	1	0	0.04	0	1.0625	0	0	0.0425	0	0	0.9575	0	0.0073	0.00699
0	0	1		0	0	1.0625	0	0	0.0425	0	0	0.9575	0.0073	0.00699

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I+pA			$\frac{1}{2}pA(I+pA)$			U_H	$\frac{1}{2}pA(I+pA)U_H$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	-0.2	0	0.02125	-0.1	0	0	0
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	-0.2125	1	0	-0.10625	0.02125	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0

$\frac{1}{2}p$ = 0.1

A			$\frac{1}{2}pA$			I			pA			I-pA			$\frac{1}{2}pA(I-pA)$			U_B	$\frac{1}{2}pA(I-pA)U_B$
0	-1	0	0	-0.1	0	1	0	0	0	-0.2	0	1	0.2	0	-0.02125	-0.1	0	0.0458	-0.000973
-1.0625	0	0	-0.10625	0	0	0	1	0	-0.2125	0	0	0.2125	1	0	-0.10625	-0.02125	0	0	-0.004866
0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0.0086	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I+pB			$\frac{1}{2}pB(I+pB)$			U_F	$\frac{1}{2}pB(I+pB)U_F$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	-0.2	0.02125	0	-0.1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	-0.2125	0	1	-0.10625	0	0.02125	0	0

$\frac{1}{2}p$ = 0.1

B			$\frac{1}{2}pB$			I			pB			I-pB			$\frac{1}{2}pB(I-pB)$			U_D	$\frac{1}{2}pB(I-pB)U_D$
0	0	-1	0	0	-0.1	1	0	0	0	0	-0.2	1	0	0.2	-0.02125	0	-0.1	0.0454	-0.000965
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.0086	0
-1.0625	0	0	-0.10625	0	0	0	0	1	-0.2125	0	0	0.2125	0	1	-0.10625	0	-0.02125	0	-0.004824

$\frac{1}{8}p^2$ = 0.005

AB			BA			$\frac{1}{8}p^2(AB+BA)$			U_J	U_G	U_C	U_A	$U_J-U_G-U_C+U_A$	$\frac{1}{8}p^2(AB+BA)(U_J-U_G-U_C+U_A)$
0	0	0	0	0	0	0	0	0	0	0	0	0.0613	0.0613	0
0	0	1.0625	0	0	0	0	0	0.0053125	0	0	0	0	0	0
0	0	0	0	1.0625	0	0	0.0053125	0	0	0	0	0	0	0

$U_{3,3} =$

0.03259
0.01186
0.01181

A numerical manipulation of FTCS for dispersion model

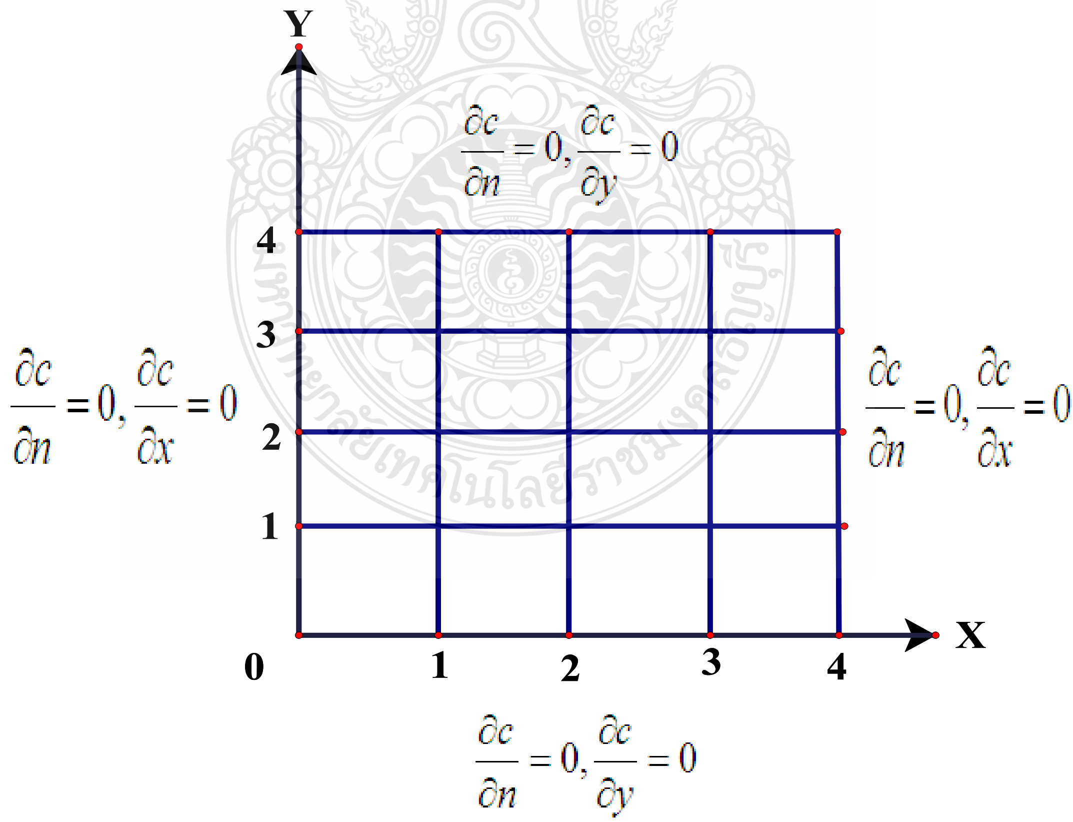
Initial Concentration

$$C(x, y, 0) = x^2(1 - \frac{x^2}{2})y^2(1 - \frac{y^2}{2})$$
$$x = l\Delta x, y = m\Delta y$$

$$\Delta x = \Delta y = 0.25$$

Table 1 Initial Concentration t=0

y, x (m)	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.003666	0.013245	0.024479	0
0.50	0	0.013245	0.047852	0.088440	0
0.75	0	0.024479	0.088440	0.163456	0
1	0	0	0	0	0



Initial Condition at t=0

$$C(x,y,0)=g(x,y)=x^2(1-\frac{x^2}{2})y^2(1-\frac{y^2}{2})$$

$$C(x,y,0)=x^2(1-\frac{x^2}{2})y^2(1-\frac{y^2}{2})$$
$$x=l\Delta x,\ y=m\Delta y$$

$$\Delta x=\Delta y=0.25$$

	l	$x=l\Delta x$	x^2	$\frac{x^2}{2}$	$(1-\frac{x^2}{2})$	m	$y=m\Delta y$	y^2	$\frac{y^2}{2}$	$(1-\frac{y^2}{2})$	$C(x,y,0)=x^2(1-\frac{x^2}{2})y^2(1-\frac{y^2}{2})$
$C_{0,0}^0$	0	0	0.00000	0.00000	1.000000	0	0	0.00000	0.00000	1.000000	**0**
$C_{0,1}^0$	0	0	0.00000	0.00000	1.000000	1	0.25	0.06250	0.03125	0.968750	**0**
$C_{0,2}^0$	0	0	0.00000	0.00000	1.000000	2	0.5	0.25000	0.12500	0.875000	**0**
$C_{0,3}^0$	0	0	0.00000	0.00000	1.000000	3	0.75	0.56250	0.28125	0.718750	**0**
$C_{0,4}^0$	0	0	0.00000	0.00000	1.000000	4	1	1.00000	0.50000	0.500000	**0**
$C_{1,0}^0$	1	0.25	0.06250	0.03125	0.968750	0	0	0.00000	0.00000	1.000000	**0**
$C_{1,1}^0$	1	0.25	0.06250	0.03125	0.968750	1	0.25	0.06250	0.03125	0.968750	**0.003665924**
$C_{1,2}^0$	1	0.25	0.06250	0.03125	0.968750	2	0.5	0.25000	0.12500	0.875000	**0.013244629**
$C_{1,3}^0$	1	0.25	0.06250	0.03125	0.968750	3	0.75	0.56250	0.28125	0.718750	**0.024478912**
$C_{1,4}^0$	1	0.25	0.06250	0.03125	0.968750	4	1	1.00000	0.50000	0.500000	**0**
$C_{2,0}^0$	2	0.5	0.25000	0.12500	0.875000	0	0	0.00000	0.00000	1.000000	**0**
$C_{2,1}^0$	2	0.5	0.25000	0.12500	0.875000	1	0.25	0.06250	0.03125	0.968750	**0.013244629**
$C_{2,2}^0$	2	0.5	0.25000	0.12500	0.875000	2	0.5	0.25000	0.12500	0.875000	**0.047851563**
$C_{2,3}^0$	2	0.5	0.25000	0.12500	0.875000	3	0.75	0.56250	0.28125	0.718750	**0.088439941**
$C_{2,4}^0$	2	0.5	0.25000	0.12500	0.875000	4	1	1.00000	0.50000	0.500000	**0**
$C_{3,0}^0$	3	0.75	0.56250	0.28125	0.718750	0	0	0.00000	0.00000	1.000000	**0**
$C_{3,1}^0$	3	0.75	0.56250	0.28125	0.718750	1	0.25	0.06250	0.03125	0.968750	**0.024478912**
$C_{3,2}^0$	3	0.75	0.56250	0.28125	0.718750	2	0.5	0.25000	0.12500	0.875000	**0.088439941**
$C_{3,3}^0$	3	0.75	0.56250	0.28125	0.718750	3	0.75	0.56250	0.28125	0.718750	**0.163455963**
$C_{3,4}^0$	3	0.75	0.56250	0.28125	0.718750	4	1	1.00000	0.50000	0.500000	**0**
$C_{4,0}^0$	4	1	1.00000	0.50000	0.500000	0	0	0.00000	0.00000	1.000000	**0**
$C_{4,1}^0$	4	1	1.00000	0.50000	0.500000	1	0.25	0.06250	0.03125	0.968750	**0**
$C_{4,2}^0$	4	1	1.00000	0.50000	0.500000	2	0.5	0.25000	0.12500	0.875000	**0**
$C_{4,3}^0$	4	1	1.00000	0.50000	0.500000	3	0.75	0.56250	0.28125	0.718750	**0**
$C_{4,4}^0$	4	1	1.00000	0.50000	0.500000	4	1	1.00000	0.50000	0.500000	**0**

Table 2 : Concentration at t=0.05

$\Delta x = \Delta y = 0.25$ $\Delta t = 0.05$

y, x (m)	0	0.25	0.50	0.75	1
0	0.000881	0.000881	0.003208	0.005918	0.005918
0.25	0.000886	0.006546	0.018874	0.025562	0.005918
0.50	0.003208	0.018874	0.050723	0.060255	0.021464
0.75	0.005919	0.025565	0.060255	0.049394	0.039603
1	0.005919	0.005919	0.021464	0.039603	0.039603

Concentration at t=0.05 use velocity u and v from

lax-wendroff method at t=0.05

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 0, m = 0$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

		$u_{0,0}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,0}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{0,0}^{n}$	$\left(0.24-0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{0,0}^{n}$	$\left(0.24-0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
	0.24	0.000000	0.000000
0			

$C_{0,0}^{1} =$ 0.000000

$v_{0,0}^{n}$	$\left(0.24+0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$$n = 0, l = 0, m = 1$$

		$u_{0,1}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,1}^{1}$	=	-0.01565	0.238435	0	0		0.000000	0.000000

$u_{0,1}^{n}$	$\left(0.24-0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.01565	0.241565	0.003666	0.000886

$v_{0,1}^{n}$	$\left(0.24-0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,1}^{1} =$ 0.000886

$v_{0,1}^{n}$	$\left(0.24+0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 0, m = 2$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t=0.05

		$u_{0,2}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,2}^{1}$	=	-0.02191	0.237809	0	0		0.000000	0.000000

$u_{0,2}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
-0.02191	0.242191	0.013245	0.003208

$v_{0,2}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
0	0.24	0.000000	0.000000	$C_{0,2}^{1} =$	0.003208

$v_{0,2}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$n = 0, l = 0, m = 3$

		$u_{0,3}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,3}^{1}$	=	-0.01784	0.238216	0	0		0.000000	0.000000

$u_{0,3}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
-0.01784	0.241784	0.024479	0.005919

$v_{0,3}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
0	0.24	0.000000	0.000000	$C_{0,3}^{1} =$	0.005919

$v_{0,3}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 0, m = 4$

u,v from lax-wendroff at t=0.05

$u = U\sqrt{gh}, v = V\sqrt{gh}$

	$u_{0,4}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,4}^{1}$ =	0	0.24	0	0	0.000000	0.000000

$u_{0,4}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{0,4}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,4}^{1} =$ 0.000000

$v_{0,4}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$n = 0, l = 1, m = 0$

	$u_{1,0}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{1,0}^{1}$ =	0	0.24	0	0	0.000000	0.000000

$u_{1,0}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{1,0}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
-0.00376	0.240376	0.003666	0.000881

$C_{1,0}^{1} =$ 0.000881

$v_{1,0}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.00376	0.239624	0	0

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$$n = 0, l = 1, m = 1$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

		$u^n_{1,1}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{1,1}$	=	-0.01565	0.238435	0	0		0.003666	0.000147

$u^n_{1,1}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
-0.01565	0.241565	0.013245	0.003199

$v^n_{1,1}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
-0.01565	0.241565	0.013245	0.003199

$C^1_{1,1} =$ 0.006546

$v^n_{1,1}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
-0.01565	0.238435	0	0

$$n = 0, l = 1, m = 2$$

		$u^n_{1,2}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{1,2}$	=	-0.02191	0.237809	0	0		0.013245	0.000530

$u^n_{1,2}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
-0.02191	0.242191	0.047852	0.011589

$v^n_{1,2}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.024479	0.005875

$C^1_{1,2} =$ 0.018874

$v^n_{1,2}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0.003665924	0.000879822

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m}+0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24-0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24-0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24+0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$n=0, l=1, m=3$

u,v from lax-wendroff at t=0.05

$u = U\sqrt{gh}, v = V\sqrt{gh}$

		$u^n_{1,3}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{1,3}$	=	-0.01784	0.238216	0	0		0.024479	0.000979

$u^n_{1,3}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
-0.01784	0.241784	0.088440	0.021383

$v^n_{1,3}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$		
0.01784	0.238216	0.000000	0.000000	$C^1_{1,3}=$	0.025565

$v^n_{1,3}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.01784	0.241784	0.013244629	0.003202339

$n=0, l=1, m=4$

		$u^n_{1,4}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{1,4}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{1,4}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{1,4}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$		
0.01784	0.238216	0.000000	0.000000	$C^1_{1,4}=$	0.005919

$v^n_{1,4}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.01784	0.241784	0.024478912	0.005918609

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 2, m = 0$

u,v from lax-wendroff at t=0.05

$u = U\sqrt{gh}, v = V\sqrt{gh}$

		$u_{2,0}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{2,0}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{2,0}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{2,0}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
-0.02191	0.242191	0.013245	0.003208

$C_{2,0}^{1} =$ 0.003208

$v_{2,0}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.02191	0.237809	0	0

$n = 0, l = 2, m = 1$

		$u_{2,1}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{2,1}^{1}$	=	0	0.24	0.003665924	0.000879822		0.013245	0.000530

$u_{2,1}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.024479	0.005875

$v_{2,1}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
-0.02191	0.242191	0.047852	0.011589

$C_{2,1}^{1} =$ 0.018874

$v_{2,1}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.02191	0.237809	0	0

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$n = 0, l = 2, m = 2$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t=0.05

		$u^n_{2,2}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{2,2}$	=	0	0.24	0.013244629	0.003178711		0.047852	0.001914

$u^n_{2,2}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.088440	0.021226

$v^n_{2,2}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.088440	0.021226

$C^1_{2,2} =$ 0.050723

$v^n_{2,2}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0.013244629	0.003178711

$n = 0, l = 2, m = 3$

		$u^n_{2,3}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{2,3}$	=	0	0.24	0.024478912	0.005874939		0.088440	0.003538

$u^n_{2,3}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.163456	0.039229

$v^n_{2,3}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
0.02692	0.237308	0.000000	0.000000

$C^1_{2,3} =$ 0.060255

$v^n_{2,3}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
0.02692	0.242692	0.047851563	0.011613192

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 2, m = 4$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

$C_{2,4}^{1}$ =

$u_{2,4}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
0	0.24	0	0	0.000000	0.000000

$u_{2,4}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{2,4}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0.02692	0.237308	0.000000	0.000000

$C_{2,4}^{1} =$ 0.021464

$v_{2,4}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0.02692	0.242692	0.088439941	0.021463666

$$n = 0, l = 3, m = 0$$

$C_{3,0}^{1}$ =

$u_{3,0}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
0	0.24	0	0	0.000000	0.000000

$u_{3,0}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{3,0}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
-0.01753	0.241753	0.024479	0.005918

$C_{3,0}^{1} =$ 0.005918

$v_{3,0}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.01753	0.238247	0	0

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$$n = 0, l = 3, m = 1$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

		$u^n_{3,1}$	$\left(0.1u^n_{l,m} + 0.24\right)$	$C^n_{l-1,m}$	$\left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,1}$	=	0.01753	0.241753	0.013244629	0.003201929		0.024479	0.000979

$u^n_{3,1}$	$\left(0.24 - 0.1u^n_{l,m}\right)$	$C^n_{l+1,m}$	$+\left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m}$
0.01753	0.238247	0.000000	0.000000

$v^n_{3,1}$	$\left(0.24 - 0.1v^n_{l,m}\right)$	$C^n_{l,m+1}$	$+\left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1}$		
-0.01753	0.241753	0.088440	0.021381	$C^1_{3,1} =$	0.025562

$v^n_{3,1}$	$\left(0.24 + 0.1v^n_{l,m}\right)$	$C^n_{l,m-1}$	$+\left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$
-0.01753	0.238247	0	0

$$n = 0, l = 3, m = 2$$

		$u^n_{3,2}$	$\left(0.1u^n_{l,m} + 0.24\right)$	$C^n_{l-1,m}$	$\left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,2}$	=	0.02692	0.242692	0.047851563	0.011613192		0.088440	0.003538

$u^n_{3,2}$	$\left(0.24 - 0.1u^n_{l,m}\right)$	$C^n_{l+1,m}$	$+\left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m}$
0.02692	0.237308	0.000000	0.000000

$v^n_{3,2}$	$\left(0.24 - 0.1v^n_{l,m}\right)$	$C^n_{l,m+1}$	$+\left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1}$		
0	0.24	0.163456	0.039229	$C^1_{3,2} =$	0.060255

$v^n_{3,2}$	$\left(0.24 + 0.1v^n_{l,m}\right)$	$C^n_{l,m-1}$	$+\left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$
0	0.24	0.024478912	0.005874939

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m}+0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24-0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24-0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24+0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$$n=0, l=3, m=3$$

u,v from lax-wendroff at t=0.05

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

		$u^n_{3,3}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,3}$	=	0.02285	0.242285	0.088439941	0.021427671		0.163456	0.006538

$u^n_{3,3}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0.02285	0.237715	0.000000	0.000000

$v^n_{3,3}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0.02285	0.237715	0.000000	0.000000

$C^1_{3,3} =$ 0.049394

$v^n_{3,3}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.02285	0.242285	0.088439941	0.021427671

$$n=0, l=3, m=4$$

		$u^n_{3,4}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,4}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{3,4}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{3,4}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0.02285	0.237715	0.000000	0.000000

$C^1_{3,4} =$ 0.039603

$v^n_{3,4}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.02285	0.242285	0.163455963	0.039602928

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$$n = 0, l = 4, m = 0$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

		$u^n_{4,0}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{4,0}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{4,0}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{4,0}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.000000	0.000000

$C^1_{4,0} =$ 0.000000

$v^n_{4,0}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0	0

$$n = 0, l = 4, m = 1$$

		$u^n_{4,1}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{4,1}$	=	0.01753	0.241753	0.024478912	0.00591785		0.000000	0.000000

$u^n_{4,1}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
0.01753	0.238247	0.000000	0.000000

$v^n_{4,1}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.000000	0.000000

$C^1_{4,1} =$ 0.005918

$v^n_{4,1}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0	0

$$C_{l,m}^{n+1}=\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}+0.04C_{l,m}^{n}+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n=0, l=4, m=2$

$u=U\sqrt{gh}, v=V\sqrt{gh}$

u,v from lax-wendroff at t=0.05

		$u_{4,2}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,2}^{1}$	=	0.02692	0.242692	0.088439941	0.021463666		0.000000	0.000000

$u_{4,2}^{n}$	$\left(0.24-0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0.02692	0.237308	0.000000	0.000000

$v_{4,2}^{n}$	$\left(0.24-0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{4,2}^{1}=$ 0.021464

$v_{4,2}^{n}$	$\left(0.24+0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$n=0, l=4, m=3$

		$u_{4,3}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,3}^{1}$	=	0.02285	0.242285	0.163455963	0.039602928		0.000000	0.000000

$u_{4,3}^{n}$	$\left(0.24-0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0.02285	0.237715	0.000000	0.000000

$v_{4,3}^{n}$	$\left(0.24-0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{4,3}^{1}=$ 0.039603

$v_{4,3}^{n}$	$\left(0.24+0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1}=\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}+0.04C_{l,m}^{n}+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n=0, l=4, m=4$$

$$u=U\sqrt{gh}, v=V\sqrt{gh}$$

u,v from lax-wendroff at t=0.05

		$u_{4,4}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,4}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{4,4}^{n}$	$\left(0.24-0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{4,4}^{n}$	$\left(0.24-0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{4,4}^{1}=$ 0.000000

$v_{4,4}^{n}$	$\left(0.24+0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

Table 3 Concentration at t=0.1

$\Delta x = \Delta y = 0.25$ $\Delta t = 0.05$

y, x (m)	0	0.25	0.50	0.75	1
0	0.000887	0.000887	0.003233	0.005953	0.005953
0.25	0.000891	0.006548	0.018966	0.025708	0.005953
0.50	0.003233	0.018965	0.050719	0.060338	0.021631
0.75	0.005954	0.025711	0.060342	0.049645	0.039836
1	0.005954	0.005954	0.021631	0.039834	0.039834

Concentration at t=0.1 use velocity u and v from

lax-wendroff method at t=0.1

$$C_{l,m}^{n+1}=\left(0.1u_{l,m}^{n}+0.24\right)C_{l-1,m}^{n}+0.04C_{l,m}^{n}+\left(0.24-0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}+\left(0.24-0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}+\left(0.24+0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n=0, l=0, m=0$$

u,v from lax-wendroff at t = 0.1

$$u=U\sqrt{gh}, v=V\sqrt{gh}$$

$C_{0,0}^{1}$ =

$u_{0,0}^{n}$	$(0.1u_{l,m}^{n}+0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n}+0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
0	0.24	0	0		0.000000	0.000000

$u_{0,0}^{n}$	$(0.24-0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24-0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{0,0}^{n}$	$(0.24-0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24-0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,0}^{1}=$ 0.000000

$v_{0,0}^{n}$	$(0.24+0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24+0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$$n=0, l=0, m=1$$

$C_{0,1}^{1}$ =

$u_{0,1}^{n}$	$(0.1u_{l,m}^{n}+0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n}+0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
-0.03024	0.236976	0	0		0.000000	0.000000

$u_{0,1}^{n}$	$(0.24-0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24-0.1u_{l,m}^{n})C_{l+1,m}^{n}$
-0.03024	0.243024	0.003666	0.000891

$v_{0,1}^{n}$	$(0.24-0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24-0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,1}^{1}=$ 0.000891

$v_{0,1}^{n}$	$(0.24+0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24+0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 0, m = 2$$

u,v from lax-wendroff at t = 0.1

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

$C_{0,2}^{1}$ =

$u_{0,2}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
-0.0411	0.23589	0	0		0.000000	0.000000

$u_{0,2}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.0411	0.24411	0.013245	0.003233

$v_{0,2}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,2}^{1}$ = 0.003233

$v_{0,2}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$$n = 0, l = 0, m = 3$$

$C_{0,3}^{1}$ =

$u_{0,3}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
-0.03221	0.236779	0	0		0.000000	0.000000

$u_{0,3}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.03221	0.243221	0.024479	0.005954

$v_{0,3}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{0,3}^{1}$ = 0.005954

$v_{0,3}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 0, m = 4$

u,v from lax-wendroff at t = 0.1

$u = U\sqrt{gh}, v = V\sqrt{gh}$

		$u_{0,4}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{0,4}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{0,4}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{0,4}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
0	0.24	0.000000	0.000000	$C_{0,4}^{1} =$	0.000000

$v_{0,4}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$n = 0, l = 1, m = 0$

		$u_{1,0}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{1,0}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{1,0}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{1,0}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
-0.01885	0.241885	0.003666	0.000887	$C_{1,0}^{1} =$	0.000887

$v_{1,0}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.01885	0.238115	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 1, m = 1$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t = 0.1

$C_{1,1}^{1}$ =

$u_{1,1}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
-0.03024	0.236976	0	0	0.003666	0.000147

$u_{1,1}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.03024	0.243024	0.013245	0.003219

$v_{1,1}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
-0.003021	0.2403021	0.013245	0.003183

$C_{1,1}^{1}$ = 0.006548

$v_{1,1}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
-0.003021	0.2396979	0	0

$n = 0, l = 1, m = 2$

$C_{1,2}^{1}$ =

$u_{1,2}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+ $C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
-0.0411	0.23589	0	0	0.013245	0.000530

$u_{1,2}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.0411	0.24411	0.047852	0.011681

$v_{1,2}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0.00006	0.239994	0.024479	0.005875

$C_{1,2}^{1}$ = 0.018965

$v_{1,2}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0.00006	0.240006	0.003665924	0.000879844

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 1, m = 3$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t = 0.1

		$u_{1,3}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{1,3}^{1}$	=	-0.03221	0.236779	0	0		0.024479	0.000979

$u_{1,3}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
-0.03221	0.243221	0.088440	0.021510

$v_{1,3}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0.03221	0.236779	0.000000	0.000000

$C_{1,3}^{1} =$ 0.025711

$v_{1,3}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0.03221	0.243221	0.013244629	0.003221372

$$n = 0, l = 1, m = 4$$

		$u_{1,4}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{1,4}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{1,4}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{1,4}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$
0.03221	0.236779	0.000000	0.000000

$C_{1,4}^{1} =$ 0.005954

$v_{1,4}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0.03221	0.243221	0.024478912	0.005953785

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m} + 0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24 - 0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24 - 0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24 + 0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$$n = 0, l = 2, m = 0$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t = 0.1

		$u^n_{2,0}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{2,0}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{2,0}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{2,0}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
-0.0411	0.24411	0.013245	0.003233

$C^1_{2,0} =$ 0.003233

$v^n_{2,0}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
-0.0411	0.23589	0	0

$$n = 0, l = 2, m = 1$$

		$u^n_{2,1}$	$(0.1u^n_{l,m} + 0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m} + 0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{2,1}$	=	-0.00019	0.239981	0.003665924	0.000879752		0.013245	0.000530

$u^n_{2,1}$	$(0.24 - 0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24 - 0.1u^n_{l,m})C^n_{l+1,m}$
-0.00019	0.240019	0.024479	0.005875

$v^n_{2,1}$	$(0.24 - 0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24 - 0.1v^n_{l,m})C^n_{l,m+1}$
-0.0411	0.24411	0.047852	0.011681

$C^1_{2,1} =$ 0.018966

$v^n_{2,1}$	$(0.24 + 0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24 + 0.1v^n_{l,m})C^n_{l,m-1}$
-0.0411	0.23589	0	0

$$C^{n+1}_{l,m} = \left(0.1u^{n}_{l,m} + 0.24\right)C^{n}_{l-1,m} + 0.04C^{n}_{l,m} + \left(0.24 - 0.1u^{n}_{l,m}\right)C^{n}_{l+1,m} + \left(0.24 - 0.1v^{n}_{l,m}\right)C^{n}_{l,m+1} + \left(0.24 + 0.1v^{n}_{l,m}\right)C^{n}_{l,m-1}$$

$n = 0, l = 2, m = 2$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t = 0.1

	$u^{n}_{2,2}$	$\left(0.1u^{n}_{l,m} + 0.24\right)$	$C^{n}_{l-1,m}$	$\left(0.1u^{n}_{l,m} + 0.24\right)C^{n}_{l-1,m}$	+	$C^{n}_{l,m}$	$0.04C^{n}_{l,m}$
$C^{1}_{2,2}$ =	0.00028	0.240028	0.013244629	0.003179082		0.047852	0.001914

$u^{n}_{2,2}$	$\left(0.24 - 0.1u^{n}_{l,m}\right)$	$C^{n}_{l+1,m}$	$+\left(0.24 - 0.1u^{n}_{l,m}\right)C^{n}_{l+1,m}$
0.00028	0.239972	0.088440	0.021223

$v^{n}_{2,2}$	$\left(0.24 - 0.1v^{n}_{l,m}\right)$	$C^{n}_{l,m+1}$	$+\left(0.24 - 0.1v^{n}_{l,m}\right)C^{n}_{l,m+1}$
0.00019	0.239981	0.088440	0.021224

$C^{1}_{2,2} =$ 0.050719

$v^{n}_{2,2}$	$\left(0.24 + 0.1v^{n}_{l,m}\right)$	$C^{n}_{l,m-1}$	$+\left(0.24 + 0.1v^{n}_{l,m}\right)C^{n}_{l,m-1}$
0.00019	0.240019	0.013244629	0.003178963

$n = 0, l = 2, m = 3$

	$u^{n}_{2,3}$	$\left(0.1u^{n}_{l,m} + 0.24\right)$	$C^{n}_{l-1,m}$	$\left(0.1u^{n}_{l,m} + 0.24\right)C^{n}_{l-1,m}$	+	$C^{n}_{l,m}$	$0.04C^{n}_{l,m}$
$C^{1}_{2,3}$ =	0.00031	0.240031	0.024478912	0.005875698		0.088440	0.003538

$u^{n}_{2,3}$	$\left(0.24 - 0.1u^{n}_{l,m}\right)$	$C^{n}_{l+1,m}$	$+\left(0.24 - 0.1u^{n}_{l,m}\right)C^{n}_{l+1,m}$
0.00031	0.239969	0.163456	0.039224

$v^{n}_{2,3}$	$\left(0.24 - 0.1v^{n}_{l,m}\right)$	$C^{n}_{l,m+1}$	$+\left(0.24 - 0.1v^{n}_{l,m}\right)C^{n}_{l,m+1}$
0.04589	0.235411	0.000000	0.000000

$C^{1}_{2,3} =$ 0.060342

$v^{n}_{2,3}$	$\left(0.24 + 0.1v^{n}_{l,m}\right)$	$C^{n}_{l,m-1}$	$+\left(0.24 + 0.1v^{n}_{l,m}\right)C^{n}_{l,m-1}$
0.04589	0.244589	0.047851563	0.011703966

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 2, m = 4$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t = 0.1

		$u_{2,4}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{2,4}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{2,4}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{2,4}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$		
0.04589	0.235411	0.000000	0.000000	$C_{2,4}^{1} =$	0.021631

$v_{2,4}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
0.04589	0.244589	0.088439941	0.021631437

$$n = 0, l = 3, m = 0$$

		$u_{3,0}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)$	$C_{l-1,m}^{n}$	$\left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{3,0}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{3,0}^{n}$	$\left(0.24 - 0.1u_{l,m}^{n}\right)$	$C_{l+1,m}^{n}$	$+\left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{3,0}^{n}$	$\left(0.24 - 0.1v_{l,m}^{n}\right)$	$C_{l,m+1}^{n}$	$+\left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n}$		
-0.0319	0.24319	0.024479	0.005953	$C_{3,0}^{1} =$	0.005953

$v_{3,0}^{n}$	$\left(0.24 + 0.1v_{l,m}^{n}\right)$	$C_{l,m-1}^{n}$	$+\left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$
-0.0319	0.23681	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 3, m = 1$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t = 0.1

		$u_{3,1}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{3,1}^{1}$	=	0.03206	0.243206	0.013244629	0.003221173		0.024479	0.000979

$u_{3,1}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0.03206	0.236794	0.000000	0.000000

$v_{3,1}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
-0.0319	0.24319	0.088440	0.021508

$C_{3,1}^{1} =$ 0.025708

$v_{3,1}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
-0.0319	0.23681	0	0

$n = 0, l = 3, m = 2$

		$u_{3,2}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{3,2}^{1}$	=	0.04589	0.244589	0.047851563	0.011703966		0.088440	0.003538

$u_{3,2}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0.04589	0.235411	0.000000	0.000000

$v_{3,2}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0.00056	0.239944	0.163456	0.039220

$C_{3,2}^{1} =$ 0.060338

$v_{3,2}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0.00056	0.240056	0.024478912	0.00587631

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m}+0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24-0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24-0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24+0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$n = 0, l = 3, m = 3$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t = 0.1

		$u^n_{3,3}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,3}$	=	0.03713	0.243713	0.088439941	0.021553963		0.163456	0.006538

$u^n_{3,3}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0.03713	0.236287	0.000000	0.000000

$v^n_{3,3}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0.03697	0.236303	0.000000	0.000000

$C^1_{3,3} =$ 0.049645

$v^n_{3,3}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.03697	0.243697	0.088439941	0.021552548

$n = 0, l = 3, m = 4$

		$u^n_{3,4}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{3,4}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{3,4}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{3,4}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0.03697	0.236303	0.000000	0.000000

$C^1_{3,4} =$ 0.039834

$v^n_{3,4}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0.03697	0.243697	0.163455963	0.039833728

$$C^{n+1}_{l,m} = \left(0.1u^n_{l,m}+0.24\right)C^n_{l-1,m} + 0.04C^n_{l,m} + \left(0.24-0.1u^n_{l,m}\right)C^n_{l+1,m} + \left(0.24-0.1v^n_{l,m}\right)C^n_{l,m+1} + \left(0.24+0.1v^n_{l,m}\right)C^n_{l,m-1}$$

$n = 0, l = 4, m = 0$

$u = U\sqrt{gh}, v = V\sqrt{gh}$

u,v from lax-wendroff at t = 0.1

		$u^n_{4,0}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{4,0}$	=	0	0.24	0	0		0.000000	0.000000

$u^n_{4,0}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0	0.24	0.000000	0.000000

$v^n_{4,0}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.000000	0.000000

$C^1_{4,0} =$ 0.000000

$v^n_{4,0}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0	0

$n = 0, l = 4, m = 1$

		$u^n_{4,1}$	$(0.1u^n_{l,m}+0.24)$	$C^n_{l-1,m}$	$(0.1u^n_{l,m}+0.24)C^n_{l-1,m}$	+	$C^n_{l,m}$	$0.04C^n_{l,m}$
$C^1_{4,1}$	=	0.03206	0.243206	0.024478912	0.005953418		0.000000	0.000000

$u^n_{4,1}$	$(0.24-0.1u^n_{l,m})$	$C^n_{l+1,m}$	$+(0.24-0.1u^n_{l,m})C^n_{l+1,m}$
0.03206	0.236794	0.000000	0.000000

$v^n_{l,m}$	$(0.24-0.1v^n_{l,m})$	$C^n_{l,m+1}$	$+(0.24-0.1v^n_{l,m})C^n_{l,m+1}$
0	0.24	0.000000	0.000000

$C^1_{4,1} =$ 0.005953

$v^n_{l,m}$	$(0.24+0.1v^n_{l,m})$	$C^n_{l,m-1}$	$+(0.24+0.1v^n_{l,m})C^n_{l,m-1}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$n = 0, l = 4, m = 2$

u,v from lax-wendroff at t = 0.1

$u = U\sqrt{gh}, v = V\sqrt{gh}$

		$u_{4,2}^{n}$	$(0.1u_{l,m}^{n}+0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n}+0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,2}^{1}$	=	0.04589	0.244589	0.088439941	0.021631437		0.000000	0.000000

$u_{4,2}^{n}$	$(0.24-0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24-0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0.04589	0.235411	0.000000	0.000000

$v_{4,2}^{n}$	$(0.24-0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24-0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
0	0.24	0.000000	0.000000	$C_{4,2}^{1} =$	0.021631

$v_{4,2}^{n}$	$(0.24+0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24+0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$n = 0, l = 4, m = 3$

		$u_{4,3}^{n}$	$(0.1u_{l,m}^{n}+0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n}+0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,3}^{1}$	=	0.03713	0.243713	0.163455963	0.039836343		0.000000	0.000000

$u_{4,3}^{n}$	$(0.24-0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24-0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0.03713	0.236287	0.000000	0.000000

$v_{4,3}^{n}$	$(0.24-0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24-0.1v_{l,m}^{n})C_{l,m+1}^{n}$		
0	0.24	0.000000	0.000000	$C_{4,3}^{1} =$	0.039836

$v_{4,3}^{n}$	$(0.24+0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24+0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

$$C_{l,m}^{n+1} = \left(0.1u_{l,m}^{n} + 0.24\right)C_{l-1,m}^{n} + 0.04C_{l,m}^{n} + \left(0.24 - 0.1u_{l,m}^{n}\right)C_{l+1,m}^{n} + \left(0.24 - 0.1v_{l,m}^{n}\right)C_{l,m+1}^{n} + \left(0.24 + 0.1v_{l,m}^{n}\right)C_{l,m-1}^{n}$$

$$n = 0, l = 4, m = 4$$

$$u = U\sqrt{gh}, v = V\sqrt{gh}$$

u,v from lax-wendroff at t = 0.1

		$u_{4,4}^{n}$	$(0.1u_{l,m}^{n} + 0.24)$	$C_{l-1,m}^{n}$	$(0.1u_{l,m}^{n} + 0.24)C_{l-1,m}^{n}$	+	$C_{l,m}^{n}$	$0.04C_{l,m}^{n}$
$C_{4,4}^{1}$	=	0	0.24	0	0		0.000000	0.000000

$u_{4,4}^{n}$	$(0.24 - 0.1u_{l,m}^{n})$	$C_{l+1,m}^{n}$	$+(0.24 - 0.1u_{l,m}^{n})C_{l+1,m}^{n}$
0	0.24	0.000000	0.000000

$v_{4,4}^{n}$	$(0.24 - 0.1v_{l,m}^{n})$	$C_{l,m+1}^{n}$	$+(0.24 - 0.1v_{l,m}^{n})C_{l,m+1}^{n}$
0	0.24	0.000000	0.000000

$C_{4,4}^{1} =$ 0.000000

$v_{4,4}^{n}$	$(0.24 + 0.1v_{l,m}^{n})$	$C_{l,m-1}^{n}$	$+(0.24 + 0.1v_{l,m}^{n})C_{l,m-1}^{n}$
0	0.24	0	0

Curriculum Vitae

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